Math 210 **Fall 2020**
Homework 3 **Drew Armstrong**

Problem 1. Gaussian Elimination. Solve the following system by converting it to a matrix and then putting the matrix in Reduced Row Echelon Form:

$$\begin{cases} x + 2y + 3z = 4, \\ x + 2y + 4z = 6, \\ x + 2y + 5z = 8. \end{cases}$$

Does the solution have the expected number of dimensions? Why or why not?

Problem 2. More Gaussian Elimination. Solve the following system by converting it to a matrix and then putting the matrix in Reduced Row Echelon Form:

$$\begin{cases} x_1 + 2x_2 + x_3 + 0 + 2x_5 = 1, \\ x_1 + 2x_2 + 2x_3 + -3x_4 + 3x_5 = 1, \\ x_1 + 2x_2 + 0 + 3x_4 + 2x_5 = 3. \end{cases}$$

Does the solution have the expected number of dimensions? Why or why not?

Problem 3. Column Relations. Put the following matrix in Reduced Row Echelon Form:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 4 & 6 \\ 2 & 3 & 5 \end{pmatrix}.$$

Use your result to find a nontrivial relation among the column vectors:

$$r \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + t \begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

for some $r, s, t \in \mathbb{R}$ that are **not all zero**. [Hint: Relations among columns are not changed by row operations, so it is easier to find a relation among the columns of RREF(A).]

Problem 4. The Solution Set of a Linear System is Flat. Consider the following system of m linear equations in n unknowns, where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{a}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$:

$$\begin{cases} \mathbf{a}_1 \bullet \mathbf{x} = b_1, \\ \mathbf{a}_2 \bullet \mathbf{x} = b_2, \\ \quad \vdots \\ \mathbf{a}_m \bullet \mathbf{x} = b_m. \end{cases}$$

If $\mathbf{x} = \mathbf{p}$ and $\mathbf{x} = \mathbf{q}$ are any two points in the solution set, prove that every point of the line $\mathbf{x} = (1-t)\mathbf{p} + t\mathbf{q}$ is also in the solution set. [Hint: Assuming that $\mathbf{a}_i \bullet \mathbf{p} = b_i$ and $\mathbf{a}_i \bullet \mathbf{q} = b_i$ for all i, you are being asked to show that $\mathbf{a}_i \bullet [(1-t)\mathbf{p} + t\mathbf{q}] = b_i$ for all i.] Remark: This implies that the solution set is a d-plane in $\mathbb{R}^n$ for some d, or it is empty.

Problem 5. Orthogonal Complement of a Subspace. A *d-dimensional subspace* of $\mathbb{R}^n$ is just a d-plane in $\mathbb{R}^n$ that contains the origin. If $\mathbf{u}_1, \ldots, \mathbf{u}_d \in \mathbb{R}^n$ are independent vectors (assume $d \leq n$) then their span is a d-dimensional subspace:

$$U = \{t_1\mathbf{u}_1 + \cdots + t_d\mathbf{u}_d : t_1, \ldots, t_d \in \mathbb{R}\}.$$

We define the *orthogonal complement* of this subspace as the set of vectors that are simultaneously perpendicular to every vector in U:[1]

$$U^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{u}_i \bullet \mathbf{x} = 0 \text{ for all } i\}.$$

Explain why $U^\perp$ is an $(n-d)$-dimensional subspace of $\mathbb{R}^n$. [Hint: The set $U^\perp$ is just the solution set of the linear equations $\mathbf{u}_i \bullet \mathbf{x} = 0$ for all i. We can express this system as a $d \times (n+1)$ matrix A. Since the rows of A are independent, we know that RREF(A) will have d pivots. So how many free variables does the system have?]

Example: If $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^3$ are independent, then $U \subseteq \mathbb{R}^3$ is the plane that they span and $U^\perp \subseteq \mathbb{R}^3$ is the line that is spanned by the cross product vector $\mathbf{u}_1 \times \mathbf{u}_2$. Hence we have $\dim U + \dim U^\perp = 2 + 1 = 3$ as expected.

[1]Remark: If $\mathbf{x}$ is perpendicular to each $\mathbf{u}_i$, then it is also perpendicular to the linear combination $t_1\mathbf{u}_1 + \cdots + t_d\mathbf{u}_d$, hence it is perpendicular to every vector in U.