Math 210 **Fall 2020**
Homework 2 **Drew Armstrong**

Problem 1. Planes in Space. (Answers are not unique.)

(a) Express the plane $\mathbf{x} = (0,0,1) + s(1,1,1) + t(1,2,3)$ in the form $ax + by + cz = d$.
(b) Express the plane $x + 2y + 4z = 6$ in the form $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$.

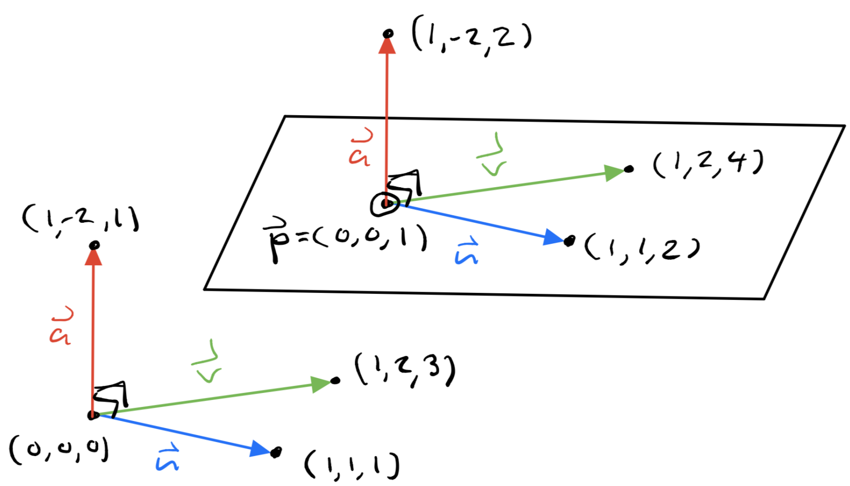
(a): We want to find an equation of the form $\mathbf{a} \bullet \mathbf{x} = d$, or $\mathbf{a} \bullet \mathbf{x} = \mathbf{a} \bullet \mathbf{p}$. Recall that this is a plane that perpendicular to the "normal vector" $\mathbf{a}$ and contains the point $\mathbf{p}$. We already have a point $\mathbf{p}$, thus we only need to find a normal vector $\mathbf{a}$. The quickest way to do this is to take the cross product of the two direction vectors $\mathbf{u} = (1,1,1)$ and $\mathbf{v} = (1,2,3)$:

$$\mathbf{a} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Thus the equation is

$$\begin{aligned} \mathbf{a} \bullet \mathbf{x} &= \mathbf{a} \bullet \mathbf{p} \\ (1,-2,1) \bullet (x,y,z) &= (1,-2,1) \bullet (0,0,1) \\ x - 2y + z &= 1. \end{aligned}$$

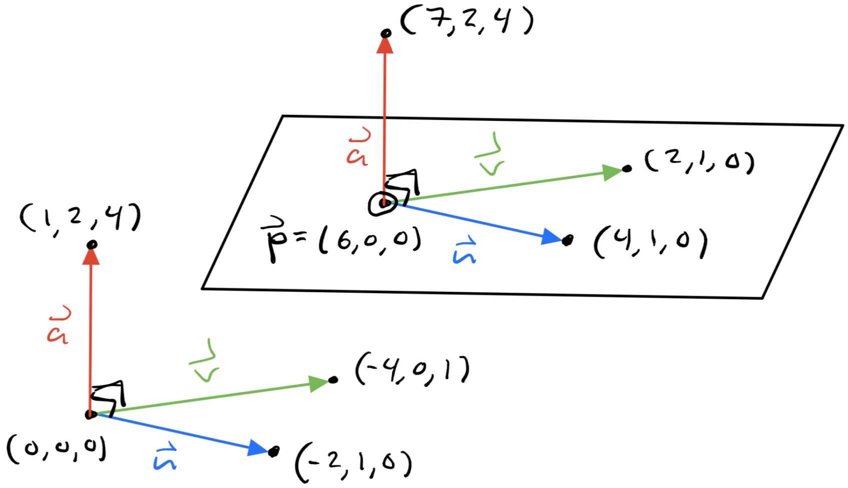
Here is a picture:



(b): To express the plane in the form $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$ we need to solve for x, y, z in terms of two parameters s and t. The easiest way to do this is to take $s = y$ and $t = z$. Then since $x + 2y + 4z = 6$ we must have $x = 6 - 2y - 4z = 6 - 2s - 4t$, and hence

$$(x,y,z) = (6 - 2s - 4t, s, t) = (6,0,0) + s(-2,1,0) + t(-4,0,1).$$

We conclude that $\mathbf{p} = (6,0,0)$ is a point on the plane, while $\mathbf{u} = (-2,1,0)$ and $\mathbf{v} = (-4,0,1)$ are two direction vectors for the plane. Here is a picture:



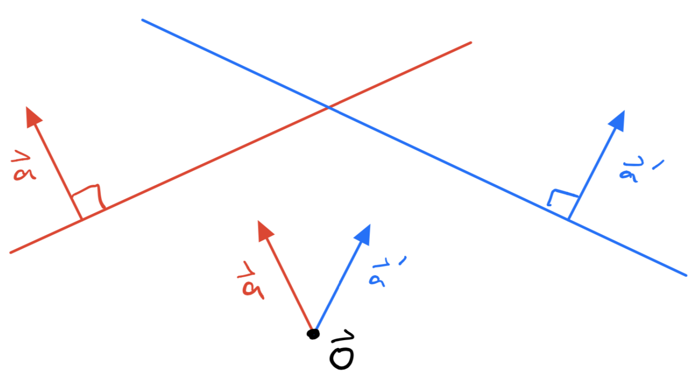
Problem 2. Perpendicular and Parallel Lines. Consider two lines in the plane:

$$ax + by = c \quad \text{and} \quad a'x + b'y = c'.$$

(a) Find an equation involving a, b, c, a', b', c' to determine when the lines are **perpendicular**. [Hint: Recall that two vectors $\mathbf{u} = (u, v)$ and $\mathbf{u}' = (u', v')$ are perpendicular if and only if $\mathbf{u} \bullet \mathbf{u} = uu' + vv' = 0$.]

(b) Find an equation involving a, b, c, a', b', c' to determine when the lines are **parallel**. [Hint: Recall that two vectors $\mathbf{u} = (u, v)$ and $\mathbf{u}' = (u', v')$ are parallel if and only if $\mathbf{u}' = t\mathbf{u}$ for some nonzero constant t.]

Remark: The two lines are perpendicular to the vectors $\mathbf{a} = (a, b)$ and $\mathbf{a}' = (a', b')$. Thus the lines are perpendicular/parallel if and only the vectors are perpendicular/parallel. The constants c and c' are irrelevant because they do not affect the slope of the lines. Picture:



(a): The lines are perpendicular when the normal vectors are perpendicular:

$$\mathbf{a} \bullet \mathbf{a}' = 0$$
$$(a, b) \bullet (a', b') = 0$$
$$aa' + bb' = 0.$$

Alternatively, the lines have slope $-a/b$ and $-a'/b'$ (assuming $b, b' \neq 0$). Hence they are perpendicular when they have "negative reciprocal slope":

$$\begin{aligned} -a/b &= b'/a' \\ -aa' &= bb' \\ aa' + bb &= 0. \end{aligned}$$

(b): The lines are parallel when the normal vectors are parallel:

$$\begin{aligned} \mathbf{a} &= t\mathbf{a}' \\ (a, b) &= (ta', tb'), \end{aligned}$$

which implies $a = ta'$ and $b = tb'$ for some t, hence $a/a' = t = b/b'$. Then we can eliminate t:

$$\begin{aligned} a/a' &= b/b' \\ ab' &= a'b \\ ab' - a'b &= 0. \end{aligned}$$

Alternatively, the lines are parallel when the slopes are equal:

$$\begin{aligned} -a/b &= -a'/b' \\ -ab' &= -a'b \\ ab' - a'b &= 0. \end{aligned}$$

See the course notes for a discussion of how this relates to the *determinant*.

Problem 3. Intersection of Two Lines. Consider the following system of two linear equations in the two unknowns x and y (where c is a constant):

$$\begin{cases} x + 3y = 6, \\ 2x + cy = 0. \end{cases}$$

(a) Solve for x and y in the case $c = -3$. Draw a picture of your solution.
(b) For which value of c does the system have **no solution**? Draw a picture in this case.

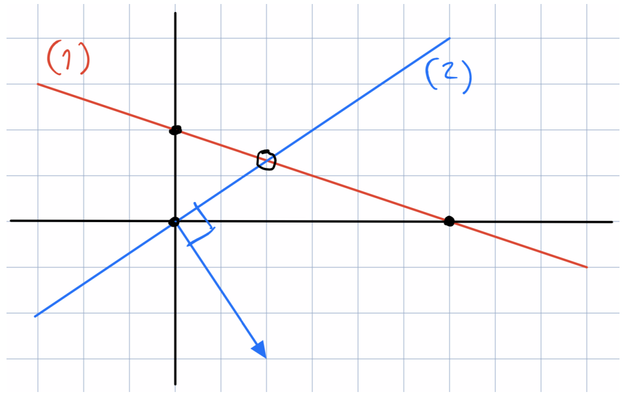
(a): Let $c = -3$ and consider the system

$$\begin{matrix} (1) \\ (2) \end{matrix} \begin{cases} x + 3y = 6, \\ 2x - 3y = 0. \end{cases}$$

Add these to obtain another true equation

$$(3) : 3x + 0y = 6.$$

This implies that $x = 2$ and then substituting $x = 2$ into either (1) or (2) gives $y = 4/3$. In other words, the two lines meet at the point $(2, 4/3)$. Picture:



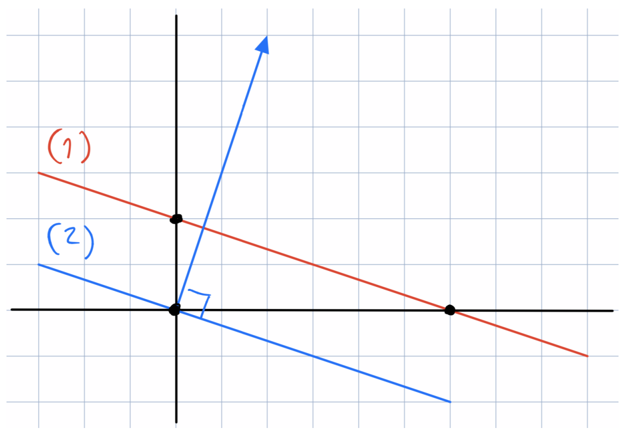
(b): Now consider the general system:

$$\begin{array}{c} (1) \\ (2) \end{array} \left\{ \begin{array}{rcrcr} x & + & 3y & = & 6, \\ 2x & + & cy & = & 0. \end{array} \right.$$

If (1) and (2) are true, then the equation (3)=2(1)-(2) is also true:

$$(3) : 0x + (6 - c)y = 12.$$

But this equation has **no solution** when $c = 6$, in which case the original system has **no solution**. Geometrically, this means that the two lines are parallel. Picture:



Remark: Changing the value of c just "rotates" the blue line (2). When $c = 6$ the blue line (2) becomes parallel to the red line (1).

Problem 4. Intersection of Two Planes (Cross Product). For any two vectors $\mathbf{u} = (u, v, w)$ and $\mathbf{u}' = (u', v', w')$ in $\mathbb{R}^3$ we define the *cross product* as follows:

$$\mathbf{u} \times \mathbf{u}' = (vw' - v'w, u'w - uw', uv' - u'v) \in \mathbb{R}^3.$$

(a) Use algebra to verify the identities $\mathbf{u} \bullet (\mathbf{u} \times \mathbf{u}') = 0$ and $\mathbf{u}' \bullet (\mathbf{u} \times \mathbf{u}') = 0$. It follows that the vector $\mathbf{u} \times \mathbf{u}'$ is simultaneously perpendicular to $\mathbf{u}$ and $\mathbf{u}'$.

(b) Use the cross product to solve the following system of linear equations:

$$\left\{ \begin{array}{rcrcrcr} x & + & y & + & 2z & = & 0, \\ 3x & + & 4y & + & 5z & = & 0. \end{array} \right.$$

[Hint: The solution is a line $(x, y, z) = t(u, v, w)$ where the vector (u, v, w) is parallel to both planes, i.e., is simultaneously perpendicular to $(1, 1, 2)$ and $(3, 4, 5)$.]

(a): Let $\mathbf{a} = \mathbf{u} \times \mathbf{u}'$. We want to check that (1) $\mathbf{u} \bullet \mathbf{a} = 0$ and (2) $\mathbf{u}' \bullet \mathbf{a} = 0$. For (1) we have

$$\begin{aligned}\mathbf{u} \bullet \mathbf{a} &= u(vw' - v'w) + v(u'w - uw') + w(uv' - u'v)\\ &= uvw' - uv'w + u'vw - uvw' + uv'w - u'vw\\ &= 0,\end{aligned}$$

and for (2) we have

$$\begin{aligned}\mathbf{u}' \bullet \mathbf{a} &= u'(vw' - v'w) + v'(u'w - uw') + w'(uv' - u'v)\\ &= u'vw' - u'v'w + u'v'w - uv'w' + uv'w' - u'vw'\\ &= 0.\end{aligned}$$

See the lecture notes for discussion of how this relates to the *determinant*.

(b): Consider the system:

$$\begin{matrix}(1)\\(2)\end{matrix}\left\{\begin{array}{rcrcrcl} x &+& y &+& 2z &=& 0,\\ 3x &+& 4y &+& 5z &=& 0.\end{array}\right.$$

We can rewrite these equations in terms of the dot product:

$$\begin{matrix}(1)\\(2)\end{matrix}\left\{\begin{array}{rcl} (1,1,2) \bullet (x,y,z) &=& 0,\\ (3,4,5) \bullet (x,y,z) &=& 0.\end{array}\right.$$

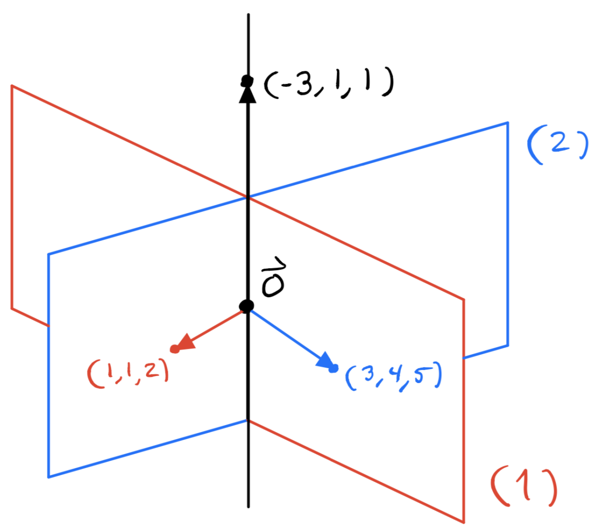
Thus we are looking for a vector $\mathbf{x} = (x,y,z)$ that is simultaneously perpendicular to $\mathbf{u} = (1,1,2)$ and $\mathbf{u}' = (3,4,5)$. From part (a) we see that the answer is given by the cross product:

$$\mathbf{x} = \mathbf{u} \times \mathbf{u}' = (1,1,2) \times (3,4,5) = (-3,1,1).$$

But this is just one point of the solution. The full solution is the whole line:

$$(x,y,z) = t(-3,1,1) = (-3t,t,t).$$

Picture:



Problem 5. Intersection of Three Planes. Consider the following system of 3 linear equations in the 3 unknowns x, y, z (where c is a constant):

$$\left\{\begin{array}{rcrcrcr} x &+& y &+& 2z &=& 0,\\ 3x &+& 4y &+& 5z &=& 0,\\ x &+& 2y &+& cz &=& -2.\end{array}\right.$$

(a) Solve for x, y, z when $c = 4$. In this case the three planes intersect at a unique point. [Hint: The intersection of the first two planes is the line $(x, y, z) = t(u, v, w)$ from 2(b). Substitute this into the third plane and solve for t.]

(b) For which value of c does the system have **no solution**? In this case the third plane is parallel to—and does not contain—the line of intersection of the first two planes. [Hint: Try to solve as in part (a). Look for a value of c that makes this impossible.]

(a): Consider the system:

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \left\{ \begin{array}{rcrcrcr} x & + & y & + & 2z & = & 0, \\ 3x & + & 4y & + & 5z & = & 0, \\ x & + & 2y & + & 4z & = & -2. \end{array} \right.$$

From Problem 4(a) we know that planes (1) and (2) meet at the line $(x, y, z) = (-3t, t, t)$. Substituting this into plane (3) gives

$$\begin{aligned} x + 2y + 4z &= -2 \\ (-3t) + 2(t) + 4(t) &= -2 \\ 3t &= -2 \\ t &= -2/3. \end{aligned}$$

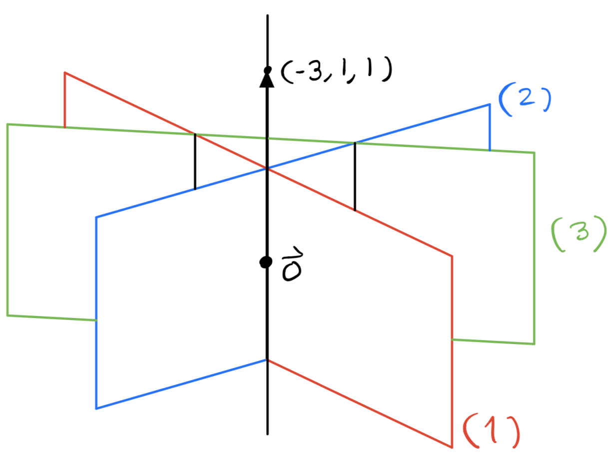
We conclude that the planes (1),(2),(3) intersect at a unique point:

$$(x, y, z) = (-3t, t, t) = (2, -2/3, -2/3).$$

(b): If we change the third plane to (3): $x + 2y + cz = -2$ for a general value of c, then the intersection of the line (1),(2) and the plane (3) is given by

$$\begin{aligned} x + 2y + cz &= -2 \\ (-3t) + 2(t) + c(t) &= -2 \\ (c - 1)t &= -2. \end{aligned}$$

If $c = 1$ then this equation has **no solution**, meaning that the line $(x, y, z) = (-3t, t, t)$ is **parallel** to the plane (3): $x + 2y + z = -2$. Then since no two of the planes (1),(2),(3) are parallel, we must have the following picture:



Remark: Changing the value of c just "rotates" the green plane (3). When $c = 1$ it becomes parallel to (and does not contain) the line of intersection of (1) and (2).