Math 210 — **Fall 2019**
Homework 6 — **Drew Armstrong**

Let A be an $n \times n$ matrix. We say that a number λ is an *eigenvalue* of A if there exists a nonzero vector $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. In this case we also say that $\mathbf{x}$ is an *eigenvector*. One can show that λ if an eigenvalue if and only if $\det(A - \lambda I) = 0$.[1]

Problem 1. Let A be a square matrix and suppose that we have $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \mu\mathbf{y}$ for some vectors $\mathbf{x}, \mathbf{y}$ and scalars λ, μ.

(a) Show that $A(s\mathbf{x}) = \lambda(s\mathbf{x})$ for all scalars s.
(b) Show that $A^n(s\mathbf{x}) = \lambda^n(s\mathbf{x})$ for all scalars s and integers $n \geq 0$.
(c) Show that $A^n(s\mathbf{x} + t\mathbf{y}) = \lambda^n(s\mathbf{x}) + \mu^n(t\mathbf{y})$ for all scalars s, t and integers $n \geq 0$.

Problem 2. Geometry of Eigenvalues.

(a) If P is any matrix satisfying $P^2 = P$ (for example, a projection), show that the only possible eigenvalues are 0 and 1. Explain this geometrically.
(b) If F is any matrix satisfying $F^2 = I$ (for example, a reflection), show that the only possible eigenvalues are $+1$ and -1. Explain this geometrically.
(c) Use the quadratic formula to show that $\lambda = \cos\theta \pm \sin\theta\sqrt{-1}$ are the eigenvalues of the rotation matrix:
$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$
For which angles θ do you get real eigenvalues? Explain this geometrically.

Problem 3. Compute all eigenvalues and eigenvectors of the following matrix:
$$A = \begin{pmatrix} 8 & -2 \\ 15 & -3 \end{pmatrix}.$$

Problem 4. A Dynamical System. Suppose that a sequence of vectors $\mathbf{x}_n = (x_n, y_n)$ is defined by the following initial condition and recurrence:
$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 30 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n/2 + y_n \\ x_n/2 \end{pmatrix} \text{ for all integers } n \geq 0.$$

(a) Express the recurrence as a matrix equation $\mathbf{x}_{n+1} = A\mathbf{x}_n$. Then the n-th vector in the sequence is given explicitly by the matrix equation $\mathbf{x}_n = A^n\mathbf{x}_0$.
(b) Verify that $\mathbf{u} = (2, 1)$ and $\mathbf{v} = (-1, 1)$ are eigenvectors of the matrix A. [One could also find these from scratch, but I decided to give you a break.]
(c) Express the initial condition $\mathbf{x}_0$ as a linear combination of eigenvectors:
$$\mathbf{x}_0 = s\mathbf{u} + t\mathbf{v} \text{ for some scalars } s \text{ and } t.$$
(d) Use 1(c) and (a) to find an explicit formula for the vector $\mathbf{x}_n = A^n(s\mathbf{u} + t\mathbf{v})$.
(e) Use your formula from part (d) to compute the limit of the sequence:
$$\lim_{n\to\infty} \mathbf{x}_n.$$

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[1] We allow the possibility that the eigenvalue λ is complex and the eigenvector $\mathbf{x}$ has complex entries.