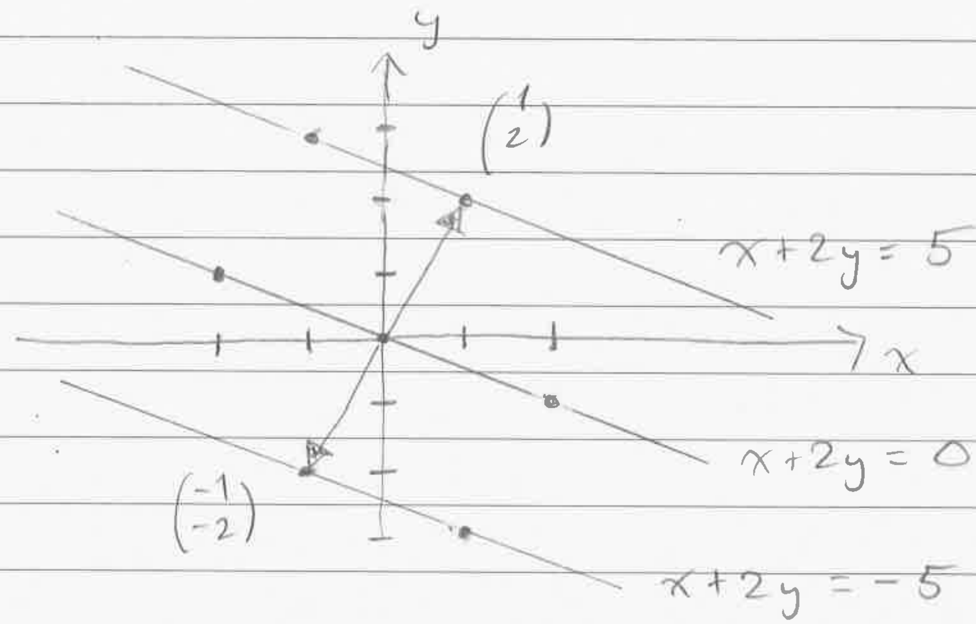
Old HW2 Solutions

Problems are slightly different

Problem 1:

(a)



(b) The equation $ax + by = c$ represents a line in the Cartesian plane that is perpendicular to the vector (a, b).

To find one specific point on the line we will intersect it with the perpendicular line $(x, y) = t(a, b) = (ta, tb)$ to get

$$ax + by = c$$
$$a(ta) + b(tb) = c$$

$$ta^2 + tb^2 = c$$

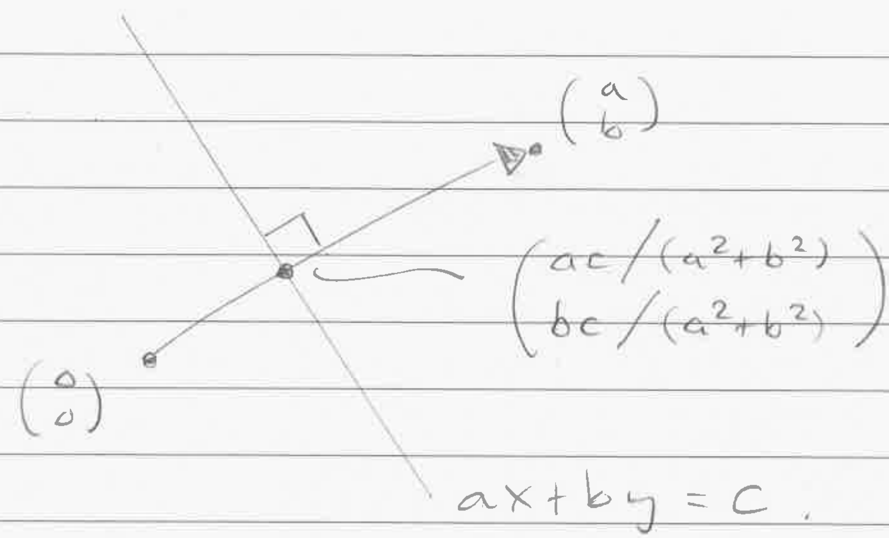
$$t(a^2+b^2) = c$$

$$t = c/(a^2+b^2).$$

This corresponds to the point

$$\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ac/(a^2+b^2) \\ bc/(a^2+b^2) \end{pmatrix}$$

Picture:



(c) The line $ax+by=c$ is $\perp$ to the vector (a,b) and the line $a'x+b'y=c'$ is $\perp$ to the vector (a',b'). This implies that the two lines are $\perp$ to each other if and only if the vectors (a, b) & (a', b') are perpendicular to each other, i.e.,

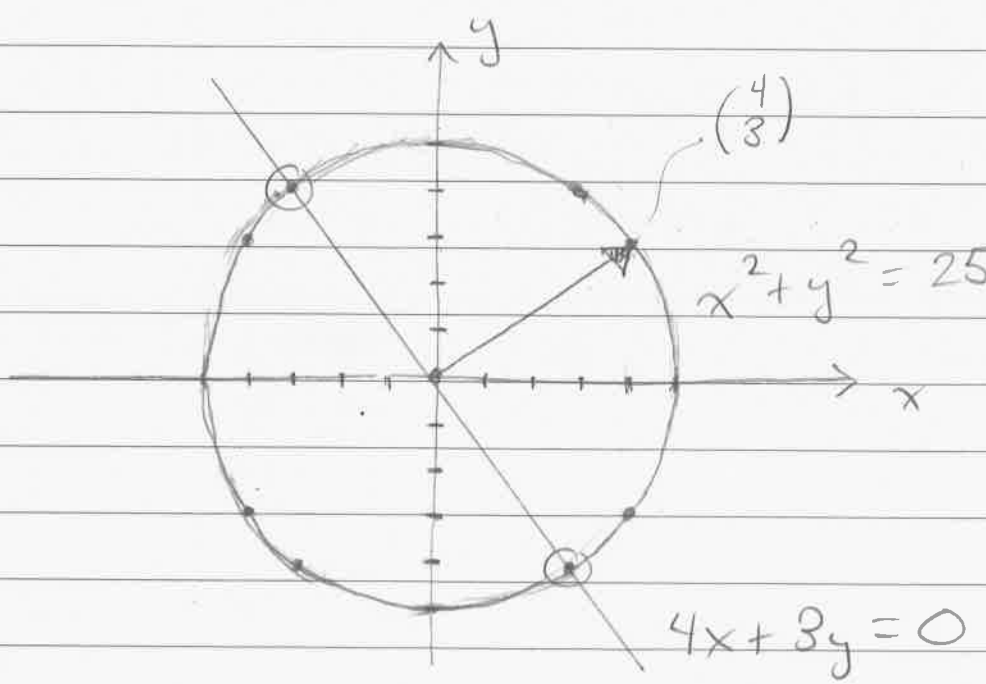
$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} a' \\ b' \end{pmatrix} = 0$$

$$\boxed{aa' + bb' = 0}$$

[Remark: This is the same answer you get using the "high-school" method of "negative reciprocal slopes". But I like this formula better because it still makes sense when one of the lines is vertical (slope ∞).]

Problem 2:

(a)



(b) We are looking for the two points of intersection as shown in the picture. To compute them we first solve

$$4x + 3y = 0$$
$$3y = -4x$$
$$y = -4/3\ x$$

and then substitute

$$x^2 + y^2 = 25$$
$$x^2 + (-4/3\ x)^2 = 25$$
$$x^2 + 16/9\ x^2 = 25$$
$$25/9\ x^2 = 25$$
$$x^2 = 9$$
$$x = \pm 3.$$

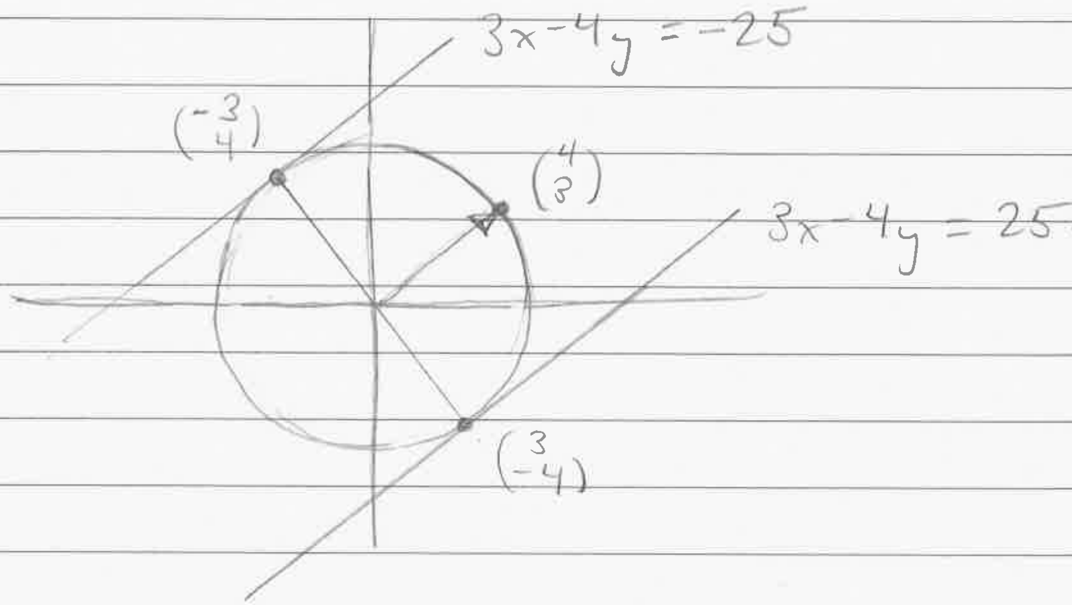
The corresponding values of y are

$$y = -4/3\,(3) = -4 \quad \& \quad y = -4/3\,(-3) = 4.$$

Thus the two points of intersection are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad \& \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}.$$

(c)



Both tangent lines are $\perp$ to the vector $(3,-4)$ so the both have on equation of the form

$$3x - 4y = c.$$

The line containing point $(-3,4)$ has

$$\begin{aligned} c &= 3x - 4y \\ &= 3(-3) - 4(4) \\ &= -9 - 16 = -25 \end{aligned}$$

and the line containing point $(3,-4)$ has

$$\begin{aligned} c &= 3x - 4y \\ &= 3(3) - 4(-4) \\ &= 9 + 16 = 25. \end{aligned}$$

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Problem 3:

(a) First we rewrite the vector equation as a system of two number equations

(*) $$x\begin{pmatrix}-1\\1\end{pmatrix}+y\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$$

$$\begin{pmatrix}-x+2y\\x+0y\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$$

(**) $$\begin{cases}-x+2y=3\\x=2\end{cases}.$$

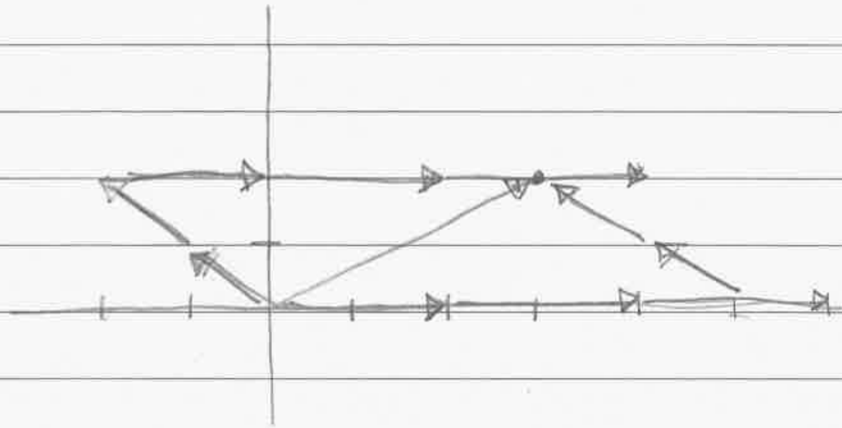
Now the solution is easy to see. Substituting $x=2$ into the first equation gives

$$-2+2y=3$$
$$2y=5$$
$$y=5/2.$$

and hence

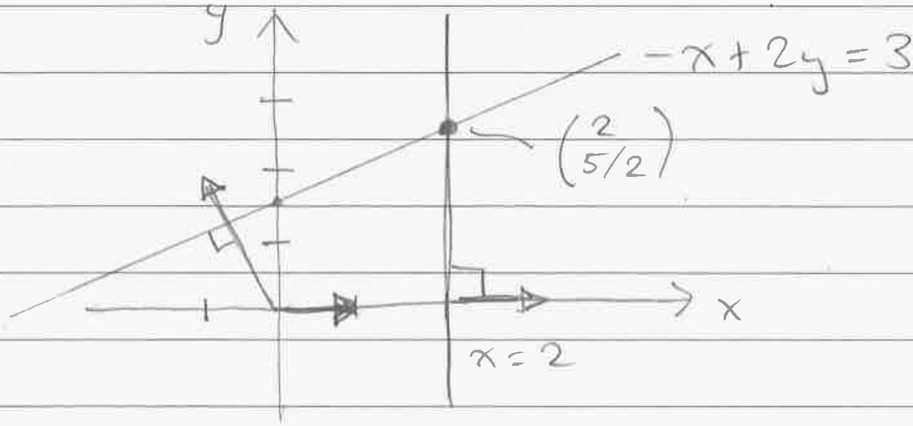
$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\\5/2\end{pmatrix}.$$

(b) Interpreting this as the solution of the vector equation (*) gives the picture:



[To get from $(0,0)$ to $(3,2)$ we move 2 times in the $(-1,1)$ direction and then $5/2$ times in the $(2,0)$ direction (or the other way around).]

(c) Interpreting this as the solution of the system (**) gives the picture:



[The two lines meet at the point $\begin{pmatrix} 2 \\ 5/2 \end{pmatrix}$.]

Problem 4 :

(a) The intersection of the planes $x+2y-z=0$ and $x+y+2z=0$ is encoded by the system of equations

$$\begin{cases} x+2y-z=0 & (1) \\ x+y+2z=0 & (2) \end{cases}$$

We can eliminate x from (1) by subtracting.

$$\begin{array}{lrcl} (1) & x+2y-z &=& 0 \\ (2) & x+y+2z &=& 0 \\ (1)-(2) & y-3z &=& 0. \quad (3) \end{array}$$

Then we can eliminate y from (2) by subtracting

$$\begin{array}{lrcl} (2) & x+y+2z &=& 0 \\ (3) & y-3z &=& 0 \\ (2)-(3) & x+5z &=& 0. \quad (4) \end{array}$$

We obtain the simpler, but equivalent, system

$$\begin{cases} x+5z=0 & (4) \\ y-3z=0 & (3) \end{cases}$$

Now letting $z = t$ be free gives the solution

$$\begin{aligned} x &= -5z = -5t \\ y &= 3z = 3t \\ z &= z = 1t \end{aligned} \implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}.$$

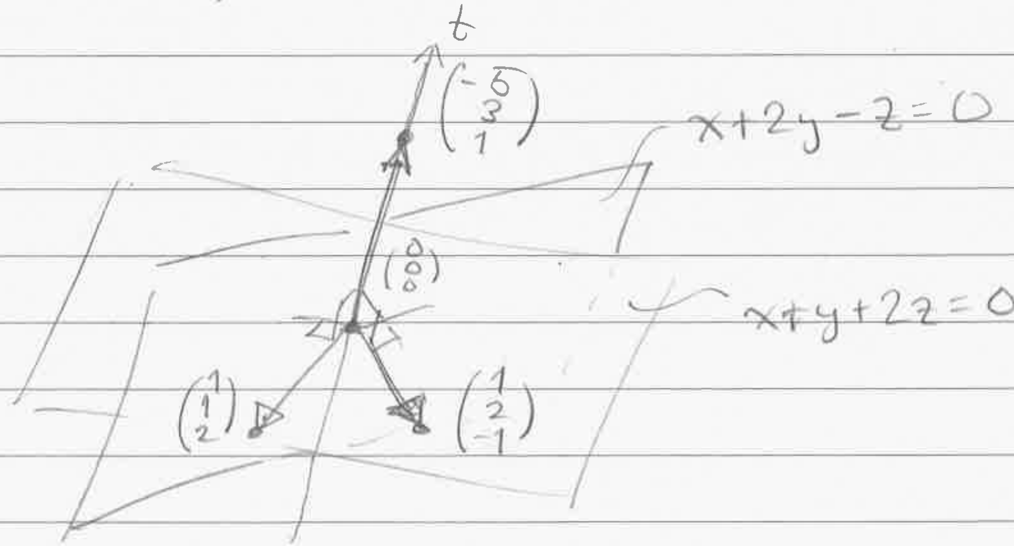
This is a <u>line</u>.

(b) Note that we can rewrite the equations (1) and (2) as

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \& \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 .$$

So we can also say that $(x, y, z) = t(-5, 3, 1)$ are precisely the vectors that are simultaneously perpendicular to both $(1, 2, -1)$ & $(1, 1, 2)$.

Picture:



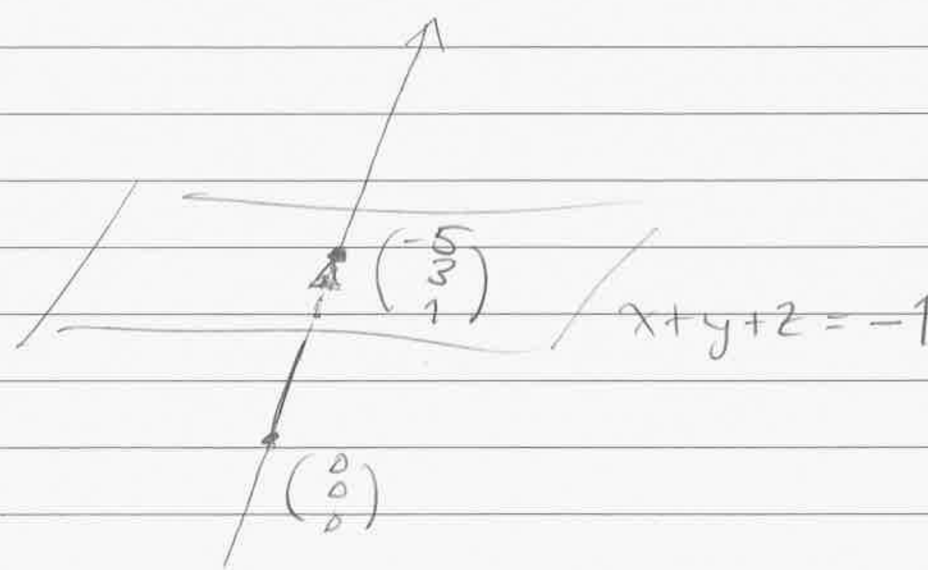
(c) Now we introduce a third plane $x+y+z=-1$. To compute the intersection of this plane with the line $(x,y,z)=(-5t,3t,t)$ we substitute to get

$$\begin{aligned} x + y + z &= -1 \\ -5t + 3t + t &= -1 \\ -t &= -1 \\ t &= 1. \end{aligned}$$

Hence the point of intersection is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}.$$

Picture:



(d) Finally, observe that the vector equation

$$x\begin{pmatrix}1\\1\\1\end{pmatrix} + y\begin{pmatrix}1\\1\\2\end{pmatrix} + z\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}-1\\0\\0\end{pmatrix}$$

is equivalent to the system of three number equations

$$\begin{cases} x + y + z = -1 \\ x + y + 2z = 0 \\ x + 2y - z = 0. \end{cases}$$

And we already solved this system in parts (a) & (c). The answer is

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-5\\3\\1\end{pmatrix}.$$

Geometrically, we interpret this as the unique point of intersection of the three planes.

Picture omitted.