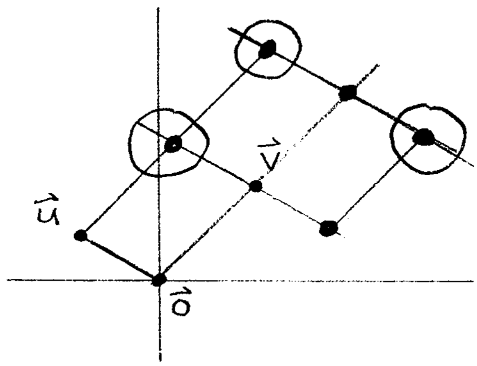
Math 210 **Fall 2019**
Exam 1 **Drew Armstrong**

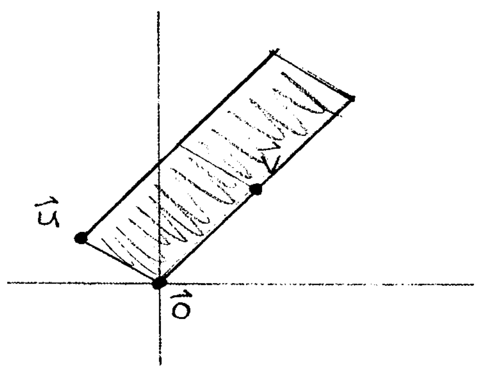
There are 6 pages, each worth 6 points, for a total of 36 points. This is a closed book test. No electronic devices are allowed.

Problem 1.

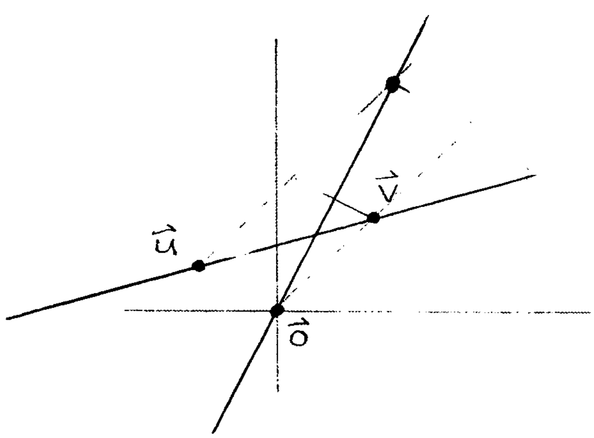
(a) Draw the points $\mathbf{u}+\mathbf{v}$, $2\mathbf{v}+\mathbf{u}$ and $2\mathbf{v}-\mathbf{u}$.



(b) Draw the shaded region $\{s\mathbf{u}+t\mathbf{v} : 0 \le s \le 1 \text{ and } 0 \le t \le 2\}$.



(c) Draw the lines $\{\mathbf{u}+t(\mathbf{v}-\mathbf{u})\}$ and $\{t(2\mathbf{v}+\mathbf{u})\}$.



Problem 2. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors with the following properties:

$$\mathbf{u} \bullet \mathbf{u} = 1, \qquad \mathbf{v} \bullet \mathbf{v} = 4 \qquad \text{and} \qquad \mathbf{u} \bullet \mathbf{v} = 1.$$

(a) Compute the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\cos\theta = \frac{u \cdot v}{\sqrt{u \cdot u}\sqrt{v \cdot v}} = \frac{1}{\sqrt{1}\sqrt{4}} = \frac{1}{2}$$

$$\theta = 60^\circ$$

(b) Compute the dot product $(\mathbf{u} + \mathbf{v}) \bullet (3\mathbf{u} - \mathbf{v})$.

$$= 3\,u \cdot u - u \cdot v + 3\,v \cdot u - v \cdot v$$

$$= 3\,u \cdot u + 2\,u \cdot v - v \cdot v$$

$$= 3(1) + 2(1) - 4 = 1$$

(c) Compute the length of $\mathbf{u} - \mathbf{v}$.

$$\|u - v\|^2 = (u - v) \cdot (u - v)$$

$$= u \cdot u - 2\,u \cdot v + v \cdot v$$

$$= 1 - 2 + 4 = 3$$

$$\|u - v\| = \sqrt{3}$$

Problem 3. Consider the plane Π defined by $x + 2y + z = 0$.

(a) Find one vector that is **perpendicular** to Π and one vector that is **parallel** to Π.

$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is perpendicular to Π.

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is parallel to Π because $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0$

(b) Compute the intersection of Π with the line $(x, y, z) = (1, 1, 0) + t(1, 0, 1)$.

$$(x,y,z) = (1+t, 1, t)$$
$$(1+t) + 2(1) + (t) = 0$$
$$3 + 2t = 0$$
$$t = -3/2$$
$$\Rightarrow (x,y,z) = \left(-\frac{1}{2}, 1, -\frac{3}{2}\right)$$

(c) Compute the intersection of Π with the plane $x + y + z = 1$.

$$\begin{cases} x + y + z = 1 \\ x + 2y + z = 0 \end{cases} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{pmatrix} \quad \text{Let } z = t$$

$$\begin{cases} x + t = 2 \\ y = -1 \end{cases} \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Problem 4. Fill in the blanks.

(a) A system of m linear equations in n unknowns represents the intersection of

m $(n-1)$ -dimensional shapes in n -dimensional space.

(b) Continuing from (a). The solution is always a flat shape. Indeed, if the two points $\mathbf{x}_0$ and $\mathbf{x}_1$ are in the solution then

every point of the form $t\vec{x}_1 + (1-t)\vec{x}_0$ is also in the solution.

(c) Continuing from (a) and (b).

If $m \leq n$ then the solution most likely has dimension $n-m$

If $m > n$ then the solution most likely does not exist

Problem 5. Consider the following system of 3 linear equations in 4 unknowns:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1, \\ 2x_1 + 2x_2 + x_3 + x_4 = 3, \\ 3x_1 + 3x_2 + 2x_3 + 2x_4 = 4. \end{cases}$$

(a) Put the system in Reduced Row Echelon Form (RREF).

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 2 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & -1 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 = 2 \\ x_3 + x_4 = -1 \\ 0 = 0 \end{cases}$$

(b) Tell me the pivot and non-pivot variables.

pivot : x_1, x_3 free : $\overset{s}{x_2}, \overset{t}{x_4}$

(c) Write down the complete solution of the system.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2-s \\ s \\ -1-t \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Problem 6. Consider the following system of 3 linear equations in 3 unknowns:

$$\begin{cases} x + y + z = 1, \\ x + 2y + 3z = 3, \\ x + y + az = b. \end{cases}$$

(a) Put the system in staircase form. [You do not need to compute the full RREF.]

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3 \\ 1 & 1 & a & b \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & a-1 & b-1 \end{pmatrix}$$

$$\begin{cases} x + y + z = 1 \\ y + 2z = 2 \\ (a-1)z = b-1 \end{cases}$$

(b) Find all the values of a and b corresponding to 0, 1 and ∞ many solutions.

There is 1 solution when $a \neq 1$

There are ∞ solutions when $a = 1 = b$

There are 0 solutions when $a = 1 \neq b$