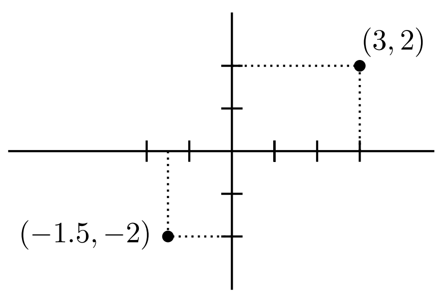
Sums of Squares (FNS 195-S) **Fall 2014**
Record of What We Did **Drew Armstrong**

Cartesian Coordinates

What is "space"? In mathematics the most basic notion of space goes back to ideas of René Descartes (1637). First of all, "space" consists of "points". In the Cartesian picture we think of points as ordered lists of real numbers.

• The symbol $\mathbb{R}^2$ denotes the set of ordered pairs (x, y) of real numbers. What does this have to do with geometry?

Consider an infinite flat plane. We will draw two imaginary "axes" (it doesn't matter where), one vertical and the other horizontal. Their point of intersection is called the "origin". Given any point P in the plane we find the closest points to P on the two axes and measure their displacements from the origin. We say that $P = (x, y)$ if the horizontal displacement is x and the vertical displacement is y. (There is no good reason for this. Once upon a time someone made a decision and now we're stuck with it. Just like Ben Franklin and the electron with "negative" charge.) The origin is $(0, 0)$.

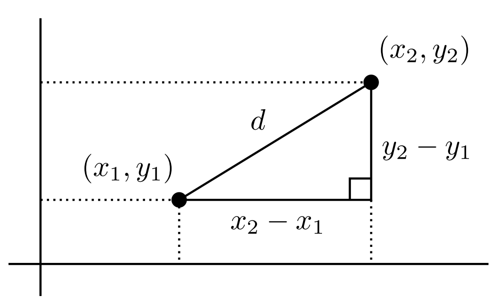


• Do the axes of the Cartesian plane need to be perpendicular?

I have two answers for this. Answer 1: How could you even tell if they weren't? Answer 2: Yes. Very shortly when we compute the distance between two points we will **assume** that the axes are perpendicular.

• What is the distance between points (x_1, y_2) and (x_2, y_2)?

Let the distance be d. We can draw a triangle by projecting onto the axes. **Because the axes are pendicular**, this is a right triangle.



Because this is a right triangle, the Pythagorean Theorem (remember?) says that

$$\boxed{d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.}$$

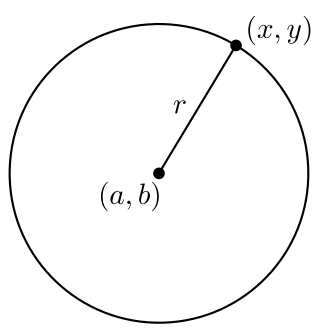
But there is a potential problem here. I drew the point (x_1, y_1) below and to the left of (x_2, y_2). What if it isn't? If (x_1, y_1) is above and to the right of (x_2, y_2) then the quantities $x_2 - x_1$ and $y_2 - y_1$ are both **negative**. If we draw the triangle in that case then the side lengths will be $-(x_2 - x_1) = x_1 - x_2$ and $-(y_2 - y_1) = y_1 - y_2$. Is that a problem? No, because luckily $(x_2 - x_1)^2 = (x_1 - x_2)^2$ and $(y_2 - y_1)^2 = (y_1 - y_2)^2$ so the same formula still works. That's convenient. We should also consider "degenerate" triangles; for example if $x_1 = x_2$ then the points lie on a vertical line. In this case the distance between them is either $y_2 - y_1$ or $y_1 - y_2$, depending on which is positive. Does that agree with our formula? Since d is positive (at least non-negative) we have

$$\begin{aligned}
d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\
d^2 &= 0^2 + (y_2 - y_1)^2 \\
d^2 &= (y_2 - y_1)^2 \\
d &= |y_2 - y_1|.
\end{aligned}$$

And luckily $|y_2 - y_1|$ equals $y_2 - y_1$ or $y_1 - y_2$, depending on which is positive. It seems that algebra is smarter than geometry. Even though there at least 9 different ways to draw the picture, the same formula holds for all of them.

• What is the equation of the **circle** with radius r centered at (a, b)?

By definition, the point (x, y) is on the circle if and only if the distance between (a, b) and (x, y) is r.



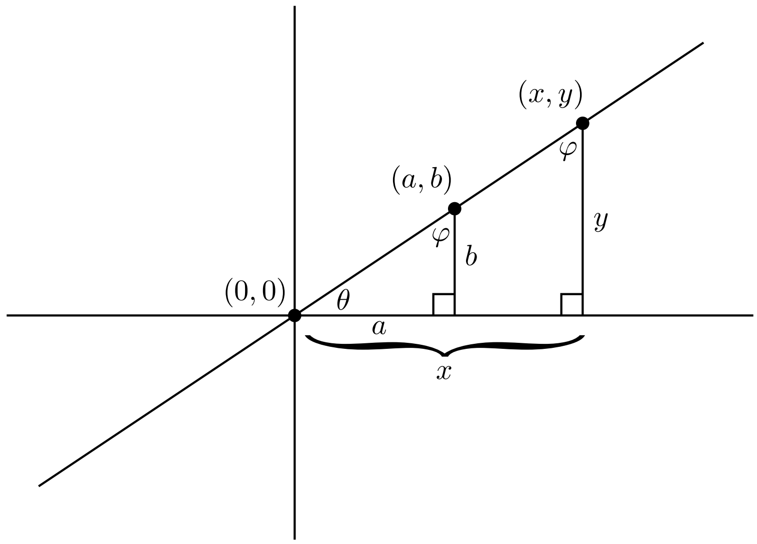
In other words, the point (x, y) is on the circle if and only if the following equation holds:

$$\boxed{(x - a)^2 + (y - b)^2 = r^2.}$$

Remark: It was Descartes' innovation to replace geometric shapes with equations. What would Euclid have thought of this?

• What is the equation of the **line** passing through the points $(0, 0)$ and (a, b)? What does it have to do with similar triangles? How do we know if two such lines are perpendicular?

If $(a, b) \neq (0, 0)$ then there exists a unique line containing the points $(0, 0)$ and (a, b). Let (x, y) be **any point** on this line. We will consider two triangles: (1) the triangle with vertices $(0, 0)$, (a, b), $(a, 0)$ and (2) the triangle with vertices $(0, 0)$, (x, y), $(x, 0)$. For example, they might look like this.



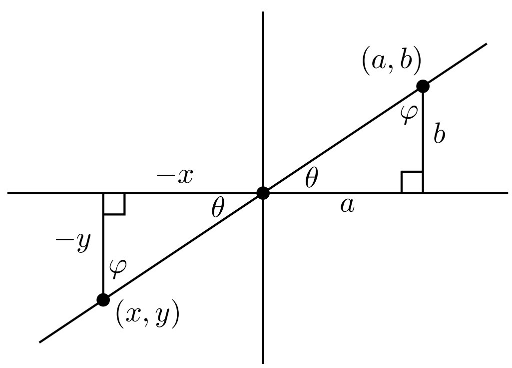
Since these two triangles are **similar** (i.e., have the same angles) we know that the corresponding side lengths have a common ratio. In particular, we have

$$\frac{a}{b} = \frac{x}{y}.$$

You might be happier writing this as

$$y = \frac{b}{a}x$$

and calling $\frac{b}{a}$ the **slope** of the line. But there is a potential problem because our diagram was not completely general. What happens if the point (a, b) is below and to the left of $(0, 0)$? Then we get a diagram like this.



Note that x and y are negative numbers here, so that $-x$ and $-y$ are **positive** numbers. We still have two similar triangles, which implies that

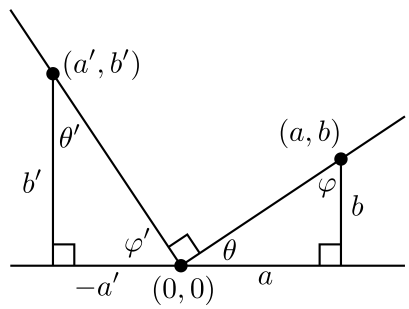
$$\frac{a}{b} = \frac{-x}{-y}.$$

Luckily this is the same equation as before. But there is one final problem. What happens if $a = 0$ or $b = 0$? If $a = 0$ then the equation of the line should be $y = 0$ and if $b = 0$ then the equation of the line should be $x = 0$. In order to accomodate all of these cases, I prefer to write the equation of the line in this form:

$$\boxed{ay = bx.}$$

Again, the algebra is smarter than the geometry because one equation works for many different pictures.

Finally, how can we tell if the lines $ay = bx$ and $a'y = b'x$ are perpendicular? We could draw them and then measure the angle with a protractor. No? Then we probably want an algebraic equation. To find the equation we will draw a picture. This picture is not completely general; I'll let you check the other cases yourself.



In this diagram note that a' is negative so that $-a'$ is positive. Also note that we have $\varphi' + 90° + \theta = 180°$ (by the assumption that the lines are perpendicular), $\theta + \varphi + 90° = 180°$, and $\theta' + \varphi' + 90° = 180°$ (angles in a triangle sum to 180°). Putting these three equations together gives $\theta' = \theta$ and $\varphi' = \varphi$. Thus the two triangles are similar and we conclude that

$$\frac{-a'}{b'} = \frac{b}{a}.$$

You might recognize this from high school in the form

$$b/a = -\frac{1}{b'/a'},$$

and you might remember the slogan "perpendicular lines have negative reciprocal slopes", but I don't like to write it this way because it doesn't make any sense when $a = 0$, $b' = 0$, or $a' = 0$. If we write the equation as

$$aa' = -bb'$$

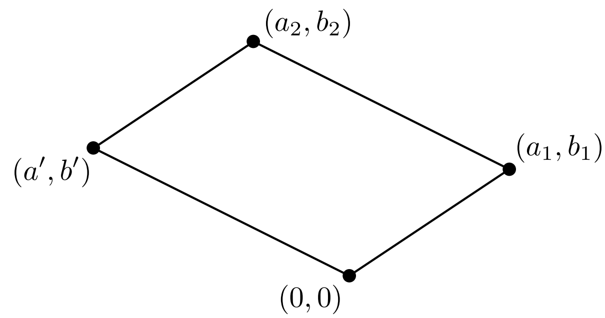
then it applies correctly in all cases. But I have a secret reason to write it like this:

$$\boxed{aa' + bb' = 0.}$$

We have just discovered the dot product, but we don't know it yet.

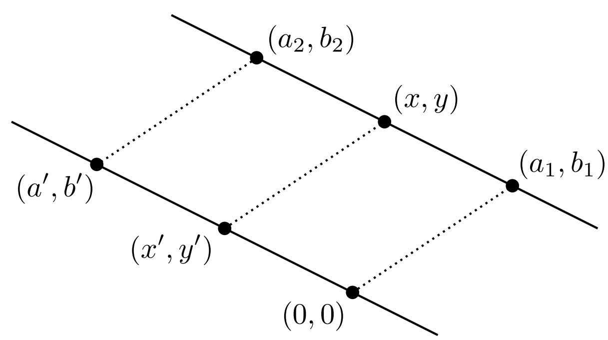
• What is the equation of the line determined by the points (a_1, b_1) and (a_2, b_2)? Why? What is the most efficient way to write the general equation of a line?

We already know the equation of a line that passes through $(0, 0)$. We should try to use this knowledge to minimize our work. If neither of (a_1, b_1) or (a_2, b_2) is equal to $(0, 0)$ then we can draw line segments from $(0, 0)$ to (a_1, b_1) and from (a_1, b_1) to (a_2, b_2). Now let (a', b') be the fourth vertex of the parallelogram determined by these line segments.



It is not difficult to see that $a' = a_2 - a_1$ and $b' = b_2 - b_1$. (Take a moment to convince yourself of this. Also try to convince yourself that this is true whether a' and b' are positive or negative. Once again, the algebra is smarter than the geometry.)

Let (x, y) be any point in the plane and define $(x', y') := (x - a_1, y - b_1)$. It is not difficult to see that the point (x, y) is on the line determined by (a_1, b_1) and (a_2, b_2) if and only if the point (x', y') is on the line determined by $(0, 0)$ and (a', b').



And we already know that the point (x', y') is on the line determined by $(0, 0)$ and (a', b') if and only if $a'y' = b'x'$. Putting it all together, we conclude that the point (x, y) is on the line determined by (a_1, b_1) and (a_2, b_2) if and only if

$$\boxed{(a_2 - a_1)(y - b_1) = (b_2 - b_1)(x - a_1).}$$

This is the equation of the line. I prefer to write it this way than, say,

$$(y - b_1) = \frac{b_2 - b_1}{a_2 - a_1}(x - a_1),$$

because this latter equation is not defined when the line is vertical (i.e., when $a_2 - a_1 = 0$). Of course, I might also like to expand everything and write the equation in this form:

$$(b_1 - b_2)x + (a_2 - a_1)y + (a_1 b_2 - a_2 b_1) = 0.$$

This has the advantage of telling me exactly when the line passes through $(0, 0)$: it happens if and only if $a_1 b_2 - a_2 b_1 = 0$.

But what if I just want to write down the equation of a general line, and I don't care which points it passes through? Then I'll write it like this:

$$\boxed{ax + by + c = 0.}$$

The numbers a, b, c are arbitrary parameters that determine the line. You might be more familiar with the point/y-intercept formula $y = mx + b$ but I don't like this because it doesn't allow us to describe vertical lines. As long as $b \neq 0$ in the equation $ax + by + c = 0$ I can always choose to put it in slope/y-intercept form if needed:

$$y = -\frac{a}{b}x - \frac{c}{b}.$$

- Compute the intersection of two general lines.

As discussed, a general line in $\mathbb{R}^2$ has the equation $ax + by + c = 0$ for some a, b, c. So let us consider two different lines

$$a_1 x + b_1 y + c_1 = 0 \tag{1}$$
$$a_2 x + b_2 y + c_2 = 0. \tag{2}$$

[Thinking Problem: Sometimes two different equations may represent the same line. For example, if $a_1 = \lambda a_2$, $b_1 = \lambda b_2$, and $c_1 = \lambda c_2$ for some $\lambda \neq 0$, then the equations (1) and (2) represent the same line. Is this the only way it can happen? Hint: Yes it is. Another Thinking Problem: What happens if $a_1 = b_1 = 0$ but $c_1 \neq 0$? Answer: Then equation (1) does **not** determine a line. Blow your mind: Putting these two thinking problems together tells us that the set of all possible lines in $\mathbb{R}^2$ can be thought of as a möbius band. (What?!)]

Normally the lines (1) and (2) will intersect at a single point (x, y). We can compute the coordinates of this point using mechanical algebraic manipulation and no (or very little) thinking. First let's assume that $a_1 \neq 0$. Then we can rearrange (1) to get

$$x = \frac{-b_1 y - c_1}{a_1}.$$

We substitute this into (2) to get

$$\begin{aligned}
a_2\left(\frac{-b_1 y - c_1}{a_1}\right) + b_2 y + c_2 &= 0 \\
a_2(-b_1 y - c_1) + a_1 b_2 y + a_1 c_2 &= 0 \\
-a_2 b_1 y - a_2 c_1 + a_1 b_2 y + a_1 c_2 &= 0 \\
(a_1 b_2 - a_2 b_1) y + (a_1 c_2 - a_2 c_1) &= 0 \\
(a_1 b_2 - a_2 b_1) y &= (a_2 c_1 - a_1 c_2) \\
y &= \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1}.
\end{aligned}$$

Then we substitute this back into (1) to get

$$\begin{aligned}
x &= -\frac{b_1}{a_1}\left(\frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right) - \frac{c_1}{a_1} \\
&= -\frac{b_1}{a_1}\left(\frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right) - \frac{c_1}{a_1}\left(\frac{a_1b_2 - a_2b_1}{a_1b_2 - a_2b_1}\right) \\
&= \frac{-b_1(a_2c_1 - a_1c_2) - c_1(a_1b_2 - a_2b_1)}{a_1(a_1b_2 - a_2b_1)} \\
&= \frac{\cancel{-a_2b_1c_1} + a_1b_1c_2 - a_1b_2c_1 + \cancel{a_2b_1c_1}}{a_1(a_1b_2 - a_2b_1)} \\
&= \frac{\cancel{a_1}(b_1c_2 - b_2c_1)}{\cancel{a_1}(a_1b_2 - a_2b_1)} \\
&= \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}.
\end{aligned}$$

Finally, we conclude that the lines (1) and (2) intersect at the single point

$$\boxed{(x, y) = \left(\frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}, \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right).}$$

This formula is called "Cramer's rule". You will learn more about it if you take a course in Linear Algebra.

The (mindless) calculation is done. Now it's time for (mindful) discussion. We began by assuming that $a_1 \neq 0$. What happens when $a_1 = 0$ (i.e., when line (1) is horizontal)? Does the same formula still hold? (Yes. Check it.) More importantly, what happens when $a_1b_2 - a_2b_1 = 0$? It seems that in this case the problem has **no solution** (because we can't divide by zero). What kind of lines never intersect? Parallel lines! This tells us that lines (1) and (2) are parallel if and only if

$$a_1b_2 - a_2b_1 = 0.$$

Does that make sense? Assuming that $b_1 \neq 0$ and $b_2 \neq 0$, the lines (1) and (2) have slopes $-a_1/b_1$ and $-a_2/b_2$. We say that the lines are parallel if they have the same slope:

$$\begin{aligned}
-a_1/b_1 &= -a_2/b_2 \\
-a_1b_2 &= -a_2b_1 \\
a_1b_2 &= a_2b_1 \\
a_1b_2 - a_1b_2 &= 0.
\end{aligned}$$

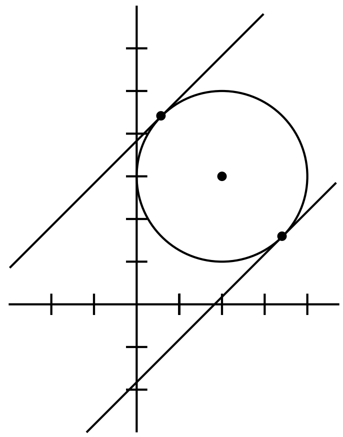
Once again algebra is smarter than geometry.

• Compute the intersection of a general line and a general circle, and the intersection of two general circles.

No. The formulas are too nasty. Let's do concrete examples instead.

• Compute the intersection of the line $y = x + c$ and the circle $(x-2)^2 + (y-3)^2 = 4$. How does the answer depend on the parameter c?

The circle has radius 2 and is centered at the point $(2, 3)$. The line $y = x + c$ has slope 1 and y-intercept c. For different values of c we get a family of parallel lines. Some of them intersect the circle, some of them don't. How does the answer depend on c? We expect that there are two special values of c, let's call them c_1 and c_2. For $c < c_1$ there will be no intersection, for $c = c_1$ the line and circle will be tangent with a unique point of intersection, for $c_1 < c < c_2$ there will two distinct points of intersection, for $c = c_2$ there will again be a unique point, and for $c > c_2$ there will be no intersection.



But how does this show up algebraically?

We substitute $y = x + c$ into the equation of the circle to get

$$\begin{aligned}(x-2)^2 + (y-3)^2 &= 4\\ (x-2)^2 + (x+c-3)^2 &= 4\\ x^2 - 4x + 4 + x^2 + 2x(c-3) + (c-3)^2 - 4 &= 0\\ 2x^2 + (2(c-3)-4)x + (c-3)^2 &= 0\\ 2x^2 + (2c-10)x + (c^2 - 6c + 9) &= 0.\end{aligned}$$

We can solve this quadratic equation for x using the quadratic formula:

$$\begin{aligned}x &= \frac{-(2c-10) \pm \sqrt{(2c-10)^2 - 4 \cdot 2(c^2 - 6c + 9)}}{2 \cdot 2}\\ &= \frac{-2c + 10 \pm \sqrt{4c^2 - 40c + 100 - 8c^2 + 48c - 72}}{4}\\ &= \frac{-2c + 10 \pm \sqrt{-4c^2 + 8c + 28}}{4}\\ &= \frac{-2c + 10 \pm \sqrt{4(-c^2 + 2c + 7)}}{4}\\ &= \frac{-2c + 10 \pm 2\sqrt{-c^2 + 2c + 7)}}{4}\\ &= \frac{-c + 5 \pm \sqrt{-c^2 + 2c + 7}}{2}.\end{aligned}$$

Then we substitute this into the equation of the line to get

$$y = \frac{-c + 5 \pm \sqrt{-c^2 + 2c + 7}}{2} + c = \frac{c + 5 \pm \sqrt{-c^2 + 2c + 7}}{2}$$

Well, that's the answer. How does it compare with the geometric picture? For different values of c the formula $\sqrt{-c^2 + 2c + 7}$ may represent 0, 1, or 2 numbers, depending on whether

$-c^2 + 4c + 7$ is negative, zero, or positive. So let's analyze this. To solve the equation $-c^2 + 2c + 7 = 0$ we again use the quadratic formula:

$$\begin{aligned} c &= \frac{-2 \pm \sqrt{4 - 4(-1)7}}{2(-1)} \\ &= \frac{-2 \pm \sqrt{32}}{-2} \\ &= \frac{-2 \pm 4\sqrt{2}}{-2} \\ &= 1 \pm 2\sqrt{2} \\ &\approx -1.83 \text{ or } 3.83 \end{aligned}$$

These are exactly the special values of c we were looking for. If $c = 1 - 2\sqrt{2} \approx -1.83$ then the line is tangent to the circle at the point

$$(x, y) = (2 + \sqrt{2}, 3 - \sqrt{2}) \approx (3.41, 1.59),$$

and if $c = 1 + 2\sqrt{2} \approx 3.83$ then the line is tangent to the circle at the point

$$(x, y) = (2 - \sqrt{2}, 3 + \sqrt{2}) \approx (0.59, 4.41).$$

Check that this agrees with the picture. If c is between -1.83 and 3.83 then there will be exactly two points of intersection given by

$$(x, y) = \left(\frac{-c + 5 + \sqrt{-c^2 + 2c + 7}}{2}, \frac{c + 5 + \sqrt{-c^2 + 2c + 7}}{2} \right)$$

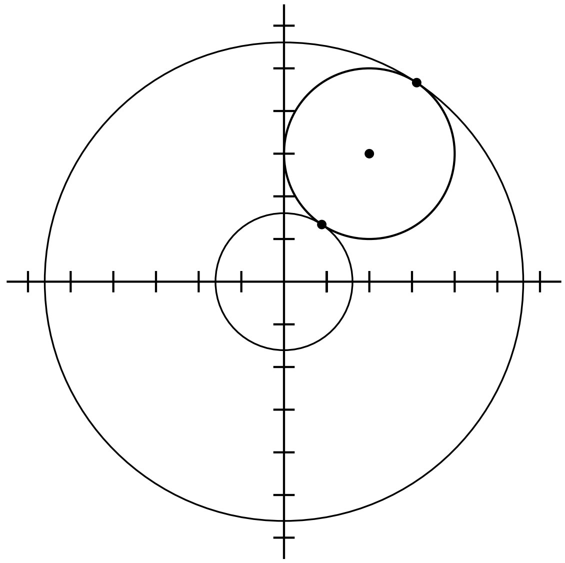
and

$$(x, y) = \left(\frac{-c + 5 - \sqrt{-c^2 + 2c + 7}}{2}, \frac{c + 5 - \sqrt{-c^2 + 2c + 7}}{2} \right).$$

If c is less than -1.83 or greater than 3.83 then there will be no intersection at all.

- Compute the intersection of the circles $x^2 + y^2 = r^2$ and $(x - 2)^2 + (y - 3)^2 = 4$. How does the answer depend on the parameter r?

Now we have replaced the line $y = x + c$ with the circle with radius r centered at $(0, 0)$. This time we expect two special values of r, let's call them r_1 and r_2 such that for r between r_1 and r_2 we get two points of intersection and for r outside of this range there is no intersection. When $r = r_1$ or $r = r_2$ the two circles will be tangent.



We expect these values of r to show up in the algebra via some function of r inside of a square root. Let's do it.

But where to begin. Last time we solved for y in the equation of then line and then substituted into the equation of the circle. If we try to emulate this we will first write $y = \pm\sqrt{r^2 - x^2}$ and then substitute to get

$$(x-2)^2 + \left(\pm\sqrt{r^2 - x^2} - 3\right)^2 = 4.$$

That looks pretty bad. We don't want to do it that way. Instead we will first expand the equation of the circle to get

$$\begin{aligned}(x-2)^2 + (y-3)^2 &= 4\\ x^2 - 4x + 4 + y^2 - 6y + 9 &= 4\\ (x^2 + y^2) - 4x - 6y + 9 &= 0\end{aligned}$$

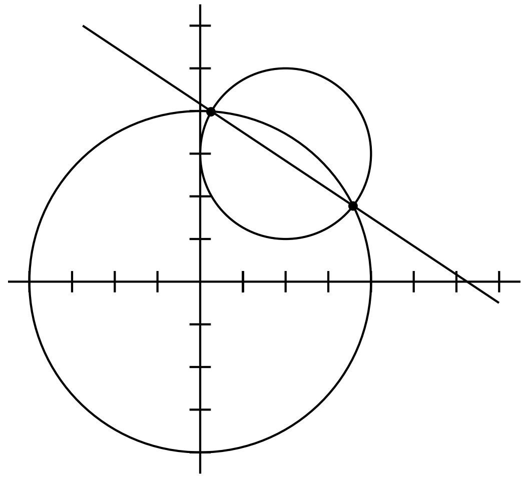
and then substitute $r^2 = x^2 + y^2$ to get

$$r^2 - 4x - 6y + 9 = 0.$$

What is this? It's the equation of a line:

$$y = -\frac{2}{3}x + \frac{9 + r^2}{6}.$$

Yeah, but what does this line have to do with the problem? Good question. Let's try an example. When $r = 4$ the two circles and the line look like this:



Aha! So this is the line that contains the two points of intersection of the circles. That means we can throw away one of the circles and just compute the intersection of a circle and a line (which we already know how to do). Which circle do you want to throw away? It doesn't matter, so I'll throw away the circle centered at $(2, 3)$. Now we substitute the equation of the line into $x^2 + y^2 = r^2$ to get

$$\begin{aligned}
x^2 + \left(-\frac{2}{3}x + \frac{9+r^2}{6}\right)^2 &= r^2 \\
x^2 + \frac{4}{9}x^2 - 2\cdot\frac{2}{3}\cdot\frac{9+r^2}{6}x + \left(\frac{9+r^2}{6}\right)^2 &= r^2 \\
\frac{13}{9}x^2 - \frac{2(9+r^2)}{9}x + \frac{(9+r^2)^2}{36} &= r^2 \\
13x^2 - 2(9+r^2)x + \frac{(9+r^2)^2}{4} &= 9r^2 \\
13x^2 - 2(9+r^2)x + \frac{81+18r^2+r^4-36r^2}{4} &= 0 \\
13x^2 - 2(9+r^2)x + \frac{r^4-18r^2+81}{4} &= 0.
\end{aligned}$$

We can solve for x using the quadratic formula and skipping a few steps:

$$x = \frac{2(9+r^2) \pm 3\sqrt{-(r^2)^2 + 34(r^2) - 81}}{26}$$

We won't bother solving for y. Note that the critical values of r occur when

$$-(r^2)^2 + 34(r^2) - 81 = 0.$$

We can solve this using the quadratic formula to get

$$\begin{aligned}
r^2 &= \frac{-34 \pm \sqrt{(34)^2 - 4(-1)(-81)}}{-2} \\
&= \frac{-34 \pm 8\sqrt{13}}{-2} \\
&= 17 \pm 4\sqrt{13} \\
&\approx 2.58 \text{ or } 31.42,
\end{aligned}$$

and since r is positive (it's a radius), this implies that the critical values are

$$r = 1.61 \text{ and } 5.61.$$

That agrees with the first picture so I assume it's correct. When $r^2 = 17 + 4\sqrt{13}$ we get the tangent point

$$(x, y) = \left(2 + \frac{4}{13}\sqrt{13}, 3 + \frac{6}{13}\sqrt{13}\right) \approx (3.11, 4.66)$$

and when $r^2 = 17 - 4\sqrt{13}$ we get the tangent point

$$(x, y) = \left(2 - \frac{4}{13}\sqrt{13}, 3 - \frac{6}{13}\sqrt{13}\right) \approx (0.89, 1.34).$$

Okay, fine. Now for the discussion. If $r^2 < (17 - 4\sqrt{13})$ or $r^2 > (17 + 4\sqrt{13})$ then the two circles don't intersect. In this case what is the geometric interpretation of the line

$$y = -\frac{2}{3}x + \frac{9 + r^2}{6} \, ?$$

It *must* have a geometric interpretation, right?