Sums of Squares (FNS 195-S) **Fall 2014**
Daily Agenda **Drew Armstrong**

Day 1: First Day of Class

1. What is the Pythagorean Theorem? (and who was Pythagoras?)
2. **Why** is the Pythagorean Theorem true? (and **is** it true?)
3. Why do the angles in a triangle sum to 180°? (and **do** they?)
4. Find a triangle whose angles do **not** sum to 180°.
5. What is the surface area of a sphere?
6. What is the formula for the area of a triangle on the surface of a sphere?

Homework for Thursday: Look up a proof of the Pythagorean Theorem. (There are many to choose from. Try to find a novel one.) Investigate what is known about Pythagoras and the Pythagoreans.

Day 2: Surface Area of a Sphere

Show and Tell:

• Did you find a proof of the Pythagorean Theorem that you like and understand? What did you learn about Pythagoras and the Pythagoreans?

Discussion:

• What is area? What is volume?

Activity:

• If you halve the dimensions of a 3D shape, what happens to the volume? Why?

• What is a *prism*? Explain why the volume of a prism is (area of base) × (length). [Cavalieri's Principle.] Can you decompose a tetrahedron into prisms? Use this to prove that the volume of the tetrahedron is 1/3 × (area of base) × (height).

• Explain why the volume of **any** cone is 1/3 × (area of base) × (height). [Hint: Approximate the cone by tetrahedra.]

• Show that the volume of **half** of a sphere of radius r is equal to **twice** the volume of a circular cone of height r and radius r. [Cavalieri's Principle again.] Conclude that the volume of the full sphere is $4\pi r^3/3$.

• Approximate a sphere of radius r by many skinny tetrahedra of height $\approx r$. Use this to compute the **surface area** of the sphere. [Hint: You know that that the **volume** of the sphere is $4\pi r^3/3$ and the **volume** of each tetrahedron is $r/3 \times$ (area of base).]

Homework for Tuesday:

• Write down your proof of the Pythagorean Theorem and bring it to me.

• Investigate Euclid of Alexandria.

Day 3: more Surface Area of a Sphere

Homework:

• Hand in your proof of the Pythagorean Theorem. What did you learn about Euclid?

Discussion:

• What is area? What is volume?

Current Goal:

• Our goal is to prove that the surface area of a sphere of radius r is $4\pi r^2$. We want to find a convincing proof that **avoids Calculus**.

Activity:

• Cavalieri's Principle and the volume of a *prism*.

• If you halve the dimensions of a 3D shape, what happens to the volume? Why?

• Euclid's proof that the volume of a tetrahedron is $1/3 \times$ (area of base) $\times$ (height): Decompose the tetrahedron into two prisms and two smaller copies of the same tetrahedron (use Zometools). Repeat. Get a nice geometric series for the volume.

• Explain why the volume of **any** cone is $1/3 \times$ (area of base) $\times$ (height). [Hint: Approximate the cone by tetrahedra.]

• Show that the volume of **half** of a sphere of radius r is equal to **twice** the volume of a circular cone of height r and radius r. [Cavalieri's Principle again.] Conclude that the volume of the full sphere is $4\pi r^3/3$.

• Approximate a sphere of radius r by many skinny tetrahedra of height $\approx r$. Use this to compute the **surface area** of the sphere. [Hint: You know that that the **volume** of the sphere is $4\pi r^3/3$ and the **volume** of each tetrahedron is $r/3 \times$ (area of base).]

Homework for Thursday:

• Find a proof of $4\pi r^2$ that uses Calculus. Be prepared to explain it.

Day 4: more Surface Area of a Sphere

Homework:

• What did you learn about Archimedes? Did you find out how to compute the surface area of a sphere using Calculus?

Goals:

• Prove that the surface area of a sphere of radius r is $4\pi r^2$, **without using Calculus**.

• Prove that the area of a triangle on a sphere of radius r with angles α, β, γ equals

$$r^2(\alpha + \beta + \gamma - \pi).$$

Activity:

• Show that the volume of **half** of a sphere of radius r is equal to **twice** the volume of a circular cone of height r and radius r. [Cavalieri's Principle again.] Conclude that the volume of the full sphere is $4\pi r^3/3$.

• Approximate a sphere of radius r by many skinny tetrahedra of height $\approx r$. Use this to compute the **surface area** of the sphere. [Hint: You know that that the **volume** of the sphere is $4\pi r^3/3$ and the **volume** of each tetrahedron is $r/3 \times$ (area of base).]

• Look at Thomas Harriot's calculation for the area of a spherical triangle. Discuss.

Homework for Tuesday:

• The surface of a 4-dimensional "hyperball" would be a 3-dimensional "hypersphere". If the hyperball has radius r, what is the 3D volume of the hypersphere? How could we tell if we were livining in a hypersphere?

• Investigate Renée Descartes. What did he do in 1637?

Day 5: Finish Talking About Spheres

Homework:

• What did you learn about Descartes? What did you learn about "3-spheres"?

Goals:

• Compute the volume of the "3-sphere" with radius r.

• Prove that the area of a triangle on a 2-sphere of radius r with angles α, β, γ equals

$$r^2(\alpha + \beta + \gamma - \pi).$$

Activity:

• What is the definition of a "straight line" in the plane? What is the definition of a "straight line" on the surface of a sphere?

• If you place three great circles on a sphere, how many regions do they cut out? What are the shapes of these regions?

• Look at Thomas Harriot's calculation for the area of a spherical triangle. Discuss.

Homework for Thursday:

• $\mathbb{R}^3$ is the name we use for 3-dimensional coordinate space. What is the equation of a sphere in $\mathbb{R}^3$? What is the equation of a plane in $\mathbb{R}^3$? What is the equation of a line in $\mathbb{R}^3$?

• Investigate Pierre de Fermat.

• Re-submit your proof of the Pythagorean Theorem.

Day 6: Cartesian Coordinates

Homework:

• What did you learn about Pierre de Fermat? What does Fermat have to do with Cartesian coordinates?

Discussion:

• What is "space"?

Activity:

• The symbol $\mathbb{R}^2$ denotes the set of ordered pairs (x, y) of real numbers. What does this have to do with geometry?

• Do the axes of the Cartesian plane need to be perpendicular? How would we know if they weren't?

• What is the distance between two points (x_1, y_1) and (x_2, y_2)? Why?

• What is the equation of a circle of radius r centered at (a, b)?

• What is the equation of the line determined by $(0, 0)$ and (a, b)? What does it have to do with similar triangles? How do we know if two such lines are perpendicular?

• What is the equation of the line determined by two points (a_1, b_1) and (a_2, b_2)?

• Write down the equations of a general line and a general circle. How can you compute their point(s) of intersection?

• Can you compute the point(s) of intersection of two circles?

• What is the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in "Cartesian space" $\mathbb{R}^3$. Why? (This is a bit harder than you might think.)

• What is the equation of a sphere in $\mathbb{R}^3$ with radius r and center (a, b, c)?

• What is the equation of a plane in $\mathbb{R}^3$? What is the equation of a line in $\mathbb{R}^3$? (These are quite a bit harder than you might think.)

Homework for Tuesday:

• Re-submit your proof of the Pythagorean Theorem if you haven't yet.

• Choose your favorite mathematician so far and write a one page summary about them.

• Think about a topic for Independent Study 1.

Day 7: more Cartesian Coordinates

Homework:

• Re-submit Pythagorean Theorem. Submit summary of a mathematician. Did you choose a topic for the independent study?

Activity:

• Recall the formula for the distance between (x_1, y_1) and (x_2, y_2). Recall the equation for the circle of radius r centered at (a, b). Recall the equation of the line determined by the points $(0, 0)$ and (a, b). Recall the formula for when two such lines are perpendicular.

• What is the equation for the line determined by points (a_1, b_1) and (a_2, b_2)? Why? What is the most efficient way to write the general equation of a line?

• Compute the intersection of two general lines.

• Compute the intersection of a general line and circle.

• Compute the intersection of two general circles.

• What is the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in "Cartesian space" $\mathbb{R}^3$? Why? (This is a bit harder than you might think.)

• What is the equation of the sphere of radius r centered at (a, b, c)?

• What is the equation of a plane in $\mathbb{R}^3$? What is the equation of a line in $\mathbb{R}^3$? (This is quite a bit harder than you might think.)

Discussion:

• What is the "distance" between two "points" (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) in "4-dimensional Cartesian space" $\mathbb{R}^4$? Why? Is this a theorem or a definition?

Homework for Thursday:

• Consider two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbb{R}^3$. If we think of them merely as lists of numbers then we might be tempted to **add** them:

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

What geometric meaning could this possibly have? Does it make any sense to "add" points?

• Read the article on the history of vectors.

Day 8: more Cartesian Coordinates

Homework:

• Did you read the article on the history of vectors?

Activity:

• Recall the equation of the line determined by points (a_1, b_1) and (a_2, b_2). Recall the general equation of a line in $\mathbb{R}^2$. Recall the formula for the intersection of two general lines.

• Compute the intersection of the line $y = x + c$ and the circle $(x - 2)^2 + (y - 3)^2 = 2^2$. How does the answer depend on the parameter c?

• Compute the intersection of the circles $(x - 2)^2 + (y - 3)^2 = 2^2$ and $x^2 + y^2 = r^2$. How does the answer depend on the parameter r?

• What is the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in "Cartesian space" $\mathbb{R}^3$? Why? (This is a bit harder than you might think.)

• What is the equation of the sphere of radius r centered at (a, b, c)?

• What is the equation of a plane in $\mathbb{R}^3$? What is the equation of a line in $\mathbb{R}^3$? (This is quite a bit harder than you might think.)

Discussion:

• What is the "distance" between two "points" (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) in "4-dimensional Cartesian space" $\mathbb{R}^4$? Why? Is this a theorem or a definition?

Homework for Next Tuesday:

• Consider two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbb{R}^3$. If we think of them merely as lists of numbers then we might be tempted to **add** them:

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

What geometric meaning could this possibly have? Does it make any sense to "add" points?

• Choose a topic for Independent Study 1 if you haven't done so yet. Try to find some good sources. Submit your topic idea and list of sources (they can be websites).

Day 9: begin Vectors?

Homework:

- Tell me your Independent Study topic and submit a list of sources.

Activity:

- How are 3-dimensional Cartesian coordinates defined?

- What is the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in Cartesian space $\mathbb{R}^3$? Why? (This is a bit harder than you might think.)

- What is the equation of the sphere of radius r centered at (a, b, c)?

- What is the equation of a plane in $\mathbb{R}^3$? What is the equation of a line in $\mathbb{R}^3$? (This is quite a bit harder than you might think.)

- What is the "distance" between two "points" (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) in "4-dimensional Cartesian space" $\mathbb{R}^4$? Why? Is this a theorem or a definition?

- We define a **vector** as an ordered pair of points in $\mathbb{R}^n$. Given two points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, we will write the corresponding vector as $[\mathbf{x}, \mathbf{y}]$ and think of this as a directed line segment with "tail" at $\mathbf{x}$ and "head" at $\mathbf{y}$. We will regard two vectors as "the same" if they have the same length and direction. Come up with an algebraic rule to determine when two vectors are equal.

Homework for Thursday:

- Re-submit your proof of the Pythagorean Theorem, if you want to.

- Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three points in $\mathbb{R}^n$. We define how to **add** the vectors $[\mathbf{u}, \mathbf{v}]$ and $[\mathbf{v}, \mathbf{w}]$:

$$[\mathbf{u}, \mathbf{v}] + [\mathbf{v}, \mathbf{w}] := [\mathbf{u}, \mathbf{w}].$$

(We say that vectors add "head-to-tail".) Does this rule also tell us how to add vectors when the heads and tails don't match? How could we do that?

Day 10: Vectors!

Discussion:

• What is a vector?

Activity:

• In mathematics we think of a **vector** as an ordered pair of points in $\mathbb{R}^n$. Given points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ we will write the corresponding vector as $[\mathbf{x}, \mathbf{y}]$ and think of this as an arrow with "tail" at $\mathbf{x}$ and "head" at $\mathbf{y}$. Find a formula for the **length** of the vector.

• We will regard two vectors as the same if they have the same length and direction. Come up with a formula to say when the vectors $[\mathbf{x}, \mathbf{y}]$ and $[\mathbf{u}, \mathbf{v}]$ are equal. [Hint: Draw a parallelogram.]

• Any vector can be put into **standard form** by moving its tail to the origin $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$. If $[\mathbf{0}, \mathbf{v}] = [\mathbf{x}, \mathbf{y}]$, find a formula for $\mathbf{v}$. (We will often write $\mathbf{v}$ and $[\mathbf{0}, \mathbf{v}]$ to refer to the same vector. That is, if we don't specify the position of the tail, it is assumed that the tail is at $\mathbf{0}$.)

• Vectors can be "added". Given three points $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ we define the sum

$$[\mathbf{u}, \mathbf{v}] + [\mathbf{v}, \mathbf{w}] := [\mathbf{u}, \mathbf{w}].$$

We say that vectors add "head-to-tail". Draw a picture of this. [Question: Why do we use the symbol "+" to refer to this operation? What does it have to do with addition of numbers?]

• Come up with a formula for adding two vectors in standard position:

$$[\mathbf{0}, \mathbf{x}] + [\mathbf{0}, \mathbf{y}] = ?$$

[Hint: Move one of the vectors.] Explain why $[\mathbf{0}, \mathbf{x}] + [\mathbf{0}, \mathbf{y}] = [\mathbf{0}, \mathbf{y}] + [\mathbf{0}, \mathbf{x}]$.

• The "zero vector" $[\mathbf{0}, \mathbf{0}]$ has a special property. What is this property?

• Draw a picture that shows how to "subtract" vectors.

Homework for next Tuesday:

• Re-submit your proof of the Pythagorean Theorem and your sketch of a mathematician.

• Make an appointment with me to discuss your topic for the Independent Study. Submit a list of possible sources if you haven't done so yet.

Day 11: more Vectors!

Homework:

• Resubmit your proof of PT and sketch of a mathematician.

Remember:

• Given two points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ we define their **sum**

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) := (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

even though this has no obvious geometric meaning.

• A **vector** is an ordered pair of points $[\mathbf{x}, \mathbf{y}]$ in $\mathbb{R}^n$. We think of this as a directed line segment with "tail" at $\mathbf{x}$ and "head" at $\mathbf{y}$. By the Pythagorean Theorem, the length $\|[\mathbf{x}, \mathbf{y}]\|$ of this line segment satisfies

$$\|[\mathbf{x}, \mathbf{y}]\|^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2.$$

• We say that the vectors $[\mathbf{x}, \mathbf{y}]$ and $[\mathbf{u}, \mathbf{v}]$ are **equal** if they have the same length and direction. We observed that $[\mathbf{x}, \mathbf{y}] = [\mathbf{u}, \mathbf{v}]$ if and only if

$$\mathbf{x} + \mathbf{v} = \mathbf{y} + \mathbf{u}.$$

• Every vector $[\mathbf{x}, \mathbf{y}]$ is equal to a unique vector in **standard position**: $[\mathbf{x}, \mathbf{y}] = [\mathbf{0}, \mathbf{y} - \mathbf{x}]$. Sometimes we will abuse notation and simply write "$[\mathbf{x}, \mathbf{y}] = \mathbf{y} - \mathbf{x}$". Please don't misunderstand this; we are **not** claiming that a vector is a point.

Activity:

• Vectors can be "added". Given three points $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ we define the sum

$$[\mathbf{u}, \mathbf{v}] + [\mathbf{v}, \mathbf{w}] := [\mathbf{u}, \mathbf{w}].$$

We say that vectors add "head-to-tail". Draw a picture of this. [Discussion: Why is this operation natural? Why do we use the symbol "+" to refer to this operation? What does it have to do with addition of numbers?]

• Come up with a formula for adding two vectors in standard position:

$$[\mathbf{0}, \mathbf{x}] + [\mathbf{0}, \mathbf{y}] = ?$$

Give a geometric and an algebraic reason why $[\mathbf{0}, \mathbf{x}] + [\mathbf{0}, \mathbf{y}] = [\mathbf{0}, \mathbf{y}] + [\mathbf{0}, \mathbf{x}]$.

• The "zero vector" $[\mathbf{0}, \mathbf{0}]$ has a special property. What is this property?

• What does it mean to "subtract" vectors? Draw a picture.

• The vectors $[\mathbf{0}, \mathbf{x}]$ and $[\mathbf{0}, \mathbf{y}]$ form two sides of a two-dimensional triangle in $\mathbb{R}^n$. What is the third side? Let θ be the angle between $[\mathbf{0}, \mathbf{x}]$ and $[\mathbf{0}, \mathbf{y}]$. What does the Pythagorean Theorem say about θ? Keep staring at this because it is very important.

Homework for Thursday:

• Look up a proof of the Law of Cosines. Be prepared to share your proof with us.

Day 12: Dot Product

Homework:

• Did you look up a proof of the Law of Cosines? Who wants to share theirs?

Important Definition:

• Given two points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, we define the **dot product** of their corresponding vectors as follows:

$$[\mathbf{0}, \mathbf{x}] \bullet [\mathbf{0}, \mathbf{y}] := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Note that the dot product of two vectors is just a number (also called a **scalar**). In *general* there is no reasonable way to multiply two vectors and get another vector. (Although there are a couple of very *special* ways to do this, which we'll see later.)

Activity:

From now on you are allowed to write $[\mathbf{0}, \mathbf{x}] = \mathbf{x}$, as long as you promise not to get confused about it. We'll indicate whether $\mathbf{x}$ is a vector or a point by calling it either a vector or a point.

• Find a formula to express the length of a vector $\mathbf{u} \in \mathbb{R}^n$ in terms of the dot product.

• Given three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, show that

$$\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}.$$

• Given vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, draw a picture of the triangle with sides $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x} - \mathbf{y}$. Label the angle between sides $\mathbf{x}$ and $\mathbf{y}$ by θ.

• Use mindless computation to show that

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 = 2\,(\mathbf{x} \bullet \mathbf{y}).$$

• On the other hand, use the Law of Cosines to argue that

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 = 2\,\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta.$$

What do you conclude from this?

Homework for next Tuesday:

• Hand in a one or two page outline for your Independent Study 1. It must be typed and include a bibliography. Remember that the final deadline is Tues, Oct 14.

Day 13: Applications of Dot Product

Homework:

• Hand in the outline and bibliography for Independent Study 1.

Recall:

• Given two vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ we define their **dot product**

$$\mathbf{x} \bullet \mathbf{y} := x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Now think of $\mathbf{x}$ and $\mathbf{y}$ as directed line segments with their tails at $\mathbf{0}$ and let θ be the angle between them. Last time we proved the remarkable formula

$$\boxed{\mathbf{x} \bullet \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta.}$$

Today we will see *why* this formula is remarkable.

Activity:

• Actually there are **two** angles between $\mathbf{x}$ and $\mathbf{y}$. Does this matter?

• Interpret the boxed formula when $\mathbf{x} = (x)$ and $\mathbf{y} = (y)$ are in $\mathbb{R}^1$.

• A molecule of methane is formed by four hydrogen atoms surrounding a central carbon atom. Let θ be the angle between any two of the hydrogen atoms. Use the boxed formula to compute θ. [Hint: Look at the Zome model.]

• Find the equation of the line in $\mathbb{R}^2$ that contains $(0, 0)$ and is perpendicular to the vector $\mathbf{n} = (a, b)$. We call $\mathbf{n}$ the "normal vector" of the line. When are two such lines perpendicular?

• Find the equation of the plane in $\mathbb{R}^3$ that contains the point $(0, 0, 0)$ and is perpendicular to the vector $\mathbf{n} = (a, b, c)$, called the "normal vector" of the plane.

• Find the equation of the plane in $\mathbb{R}^3$ that contains the point (x_0, y_0, z_0) and is perpendicular to the vector $\mathbf{n} = (a, b, c)$.

• A line in $\mathbb{R}^3$ can be expressed that the intersection of two planes. Write down the equations of two random planes and compute their intersection. What is the most natural way to describe the answer?

Homework for Tuesday October 14:

• **Class is canceled** this Thursday because I am out of town. (Sorry.)

• Hand in your Independent Study 1 on Tuesday Oct 14.

Day 14: Fresh Start

Activity:

• Accurately state the Pythagorean Theorem.

• Use the fact that $3^2 + 4^2 = 5^2$ to describe how to obtain a right angle using very primitive technology (just a rope).

• Let a, b and c be whole numbers. We say that (a, b, c) is a **Pythagorean triple** if $a^2+b^2 = c^2$. So, for example, $(3, 4, 5)$ is a Pythagorean triple. Explain why $(6, 8, 10)$ is also a Pythagorean triple. Use the same idea to find **infinitely many** Pythagorean triples.

• The Pythagorean triples in the previous problem are all essentially "the same". Try to find another Pythagorean triple that is essentially different. Is it possible to find infinitely many "different kinds" of Pythagorean triples?

• Show that for all numbers x and y we have

$$(x + y)^2 = (x - y)^2 + 4xy.$$

• Use the previous equation to find a formula for generating Pythagorean triples. [Hint: Try to find whole numbers x and y such that $4xy$ is a square. Start by assuming that x and y are themselves squares.]

• Does your formula generate **all possible** Pythagorean triples, or just some of them?

Homework for Thursday October 23

• Look up "Plimpton 322".

• Look up Diophantus of Alexandria.

Day 15: Pythagorean Triples

Homework:

• What did you learn about "Plimpton 322"?

• What did you learn about Diophantus of Alexandria?

Recall:

• Let a, b, c be positive integers. We say that (a, b, c) is a **Pythagorean triple** if $a^2 + b^2 = c^2$.

• Last time we used a trick of Euclid to show that

$$(2uvw, (u^2 - v^2)w, (u^2 + v^2)w)$$

is a Pythagorean triple for any positive integers u, v, w. This gives us a lot of Pythagorean triples, but does it give us **all** of them?

Activity:

• Explain why Pythagorean triples are (nearly) the same thing as "rational solutions" to the equation $x^2 + y^2 = 1$. By a rational solution, I mean that x and y must both be fractions.

• Consider the point $(-1, 0)$ on the unit circle $x^2 + y^2 = 1$, and consider a line of slope t through this point. Compute the coordinates of the other point where this line intersects the circle. Call these coordinates $(x(t), y(t))$.

• If t is a fraction, show that $x(t)$ and $y(t)$ are both fractions. Conversely, if $x(t)$ and $y(t)$ are both fractions, show that t is a fraction.

• The previous problems gives us an **explicit formula** for all of the "rational points" on the unit circle. Use this to give a complete description of all Pythagorean triples. [Hint: You already know the answer.]

• We call a polynomial equation a **Diophantine equation** when we are only interested in integer solutions. The problem of Pythagorean triples is the same as the Diophantine equation $x^2 + y^2 = z^2$, but there are many other examples of Diophantine equations. Try to solve this one: $x^2 + y^2 = 2z^2$.

Homework for Tuesday October 28

• Write up your favorite proof of the Law of Cosines and hand it in.

• Look up Hypatia of Alexandria.

Day 16: Rational Points on Circles

Homework:

• Hand in your favorite proof of the Law of Cosines.

• What did you learn about Hypatia of Alexandria?

Recall:

• Let a, b, c be positive integers. We say that (a, b, c) is a **Pythagorean triple** if $a^2 + b^2 = c^2$.

• We observe that Pythagorean triples are the same thing as "rational points" on the unit circle $x^2 + y^2 = 1$. Then we used a trick of Diophantus to show that the rational points (x, y) on the circle can be parametrized by a single rational number t:

$$(x(t), y(t)) = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \right).$$

As t ranges over all rational numbers, the point $(x(t), y(t))$ ranges over all rational points on the unit circle, **except** $(-1, 0)$.

Activity:

• Explain why the rational point $(-1, 0)$ corresponds to the "rational number" $t = \infty$.

• Use trial-and-error to find a solution to the Diophantine equation $a^2 + b^2 = 2c^2$.

• Can you use Diophantus' trick to come up with the **complete** solution to $a^2 + b^2 = 2c^2$? [Hint: You are looking for rational points on the circle $x^2 + y^2 = 2$. Consider the line of slope t containing your favorite rational point from the previous problem.]

• Try to justify the following statement: If a circle contains at least one rational point, then it must contain infinitely many, and we can write a nice formula for these rational points.

• Try to prove that the circle $x^2 + y^2 = 3$ contains **no rational points at all**. That is, try to prove that the Diophantine equation $a^2 + b^2 = 3c^2$ has **no solutions**. [Hint: If a is a whole number explain why a^2 must have the form $4k$ or $4k + 1$ for some whole number k. What can you say about $a^2 + b^2$ "modulo 4"?]

Homework for Thursday October 30

• We are starting to see that some integers **can** and some integers **cannot** be written as a sum of two squares. *Fermat's Christmas Theorem* gives the complete solution to this problem. (He stated it in a letter to Marin Mersenne, dated Dec 25, 1640.) Look up the statement of this theorem.

• Let d be a positive integer. For which values of d does the circle $x^2 + y^2 = d$ contain rational points?

Day 17: Fermat's Christmas Theorem

Homework:

• What does *Fermat's Christmas Theorem* say? We will be investigating this today.

Recall:

• The Diophantine equation $a^2 + b^2 = c^2$ has complete solution

$$(a, b, c) = d \cdot (2uv, u^2 - v^2, u^2 + v^2),$$

where u and v range over the integers.

• The Diophantine equation $a^2 + b^2 = 2c^2$ has complete solution

$$(a, b, c) = d \cdot (v^2 - u^2 + 2uv, u^2 - v^2 + 2uv, u^2 + v^2),$$

where u and v range over the integers.

• But the Diophantine equation $a^2 + b^2 = 3c^2$ has **no solution at all**. Why not?

Activity:

• If, hypothetically, $a^2 + b^2 = 3c^2$ **does have** a solution, explain why it must have a solution (a, b, c) in which a, b, c have no common divisor. We will show that it has no such solution.

• Suppose, hypothetically, that we have a solution $a^2 + b^2 = 3c^2$ in which a, b, c have no common divisor. Explain why at least one of a and b must be **odd**.

• If n is an even number, show that n^2 leaves remainder 0 when divided by 4.

• If n is an odd number, show that n^2 leaves remainder 1 when divided by 4. [This is a bit harder. Hint: There are two different cases.]

• If both of a and b are odd, show that $a^2 + b^2$ leaves remainder 2 when divided by 4.

• If exactly one of a and b is odd, show that $a^2 + b^2$ leaves remainder 1 when divided by 4.

• If c is any integer, show that $3c^2$ leaves remainder 0 or 3 when divided by 4.

• Put it all together to prove that $a^2 + b^2 = 3c^2$ has no solution.

• We have seen that no number of the form $3c^2$ can be a sum of two squares. Fermat's Christmas Theorem (1640) gives a complete answer to the question: which numbers can be written as a sum of two squares? The proof is too involved for us, but let's talk about it.

Homework for Tuesday November 4:

• None. Have a Happy Halloween.

Day 18: Funny Numbers

Homework:

• Did you have a Happy Halloween?

Recall:

• **Fermat's Two Squares Theorem** (also known as "Fermat's Christmas Theorem") says that an integer n can be expressed as a sum of two squares if and only if each of its prime factors of the form $p = 3 \pmod 4$ occurs with **even multiplicity**. The proof of this theorem uses a seemingly random algebraic identity that was known to Diophantus, called the "two-square identity" (also known as the **Brahmagupta-Fibonacci Identity**):

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$

This identity says that if two integers are each expressible as a sum of two squares, then so is their product. We will see that this identity holds some big secrets.

Activity:

• Verify that the two-square identity is true.

• In general it is not possible to "multiply" vectors in $\mathbb{R}^n$ to obtain another vector, but in the case of $n = 2$ there **is** a very special way to do this. Given (a, b) and (c, d) in $\mathbb{R}^2$ we define a funny multiplication rule:

$$(a, b) \times (c, d) := (ac - bd, ad + bc).$$

Verify that this multiplication respects the length of vectors:

$$\|(a, b) \times (c, d)\| = \|(a, b)\| \cdot \|(c, d)\|.$$

• Verify that multiplication is commutative:

$$(a, b) \times (c, d) = (c, d) \times (a, b).$$

• Verify that multiplication is associative:

$$(a, b) \times [(c, d) \times (e, f)] = [(a, b) \times (c, d)] \times (e, f).$$

• Verify that multiplication distributes over addition of vectors:

$$(a, b) \times [(c, d) + (e, f)] = [(a, b) \times (c, d)] + [(a, b) \times (e, f)].$$

• Which vector behaves like the number 1? Which vector behaves like the number 0?

• Given any vector $\mathbf{z} = (a, b)$ in $\mathbb{R}^2$, let $\mathbf{z}^* = (a, -b)$ be the reflection of $\mathbf{z}$ across the x-axis. What do you get when you multiply $\mathbf{z}$ and $\mathbf{z}^*$? Use the result to show how to **divide** by $\mathbf{z}$.

• We say that the operations $+$ and $\times$ define a **field structure** on $\mathbb{R}^2$. So what?

Homework for Tuesday November 11:

• Look up the Brahmagupta-Fibonacci Identity. Who were Brahmagupta and Fibonacci, and what did they have to do with this formula?

• Look up de Moivre's formula. Who was Abraham de Moivre?

Day 19: more Funny Numbers

Homework:

• What did you learn about Brahmagupta and Fibonacci?

• What did you learn about de Moivre and his formula?

Recall:

• We can define a funny multiplication operation on $\mathbb{R}^2$ by

$$(a, b) \times (c, d) := (ac - bd, ad + bc).$$

Last time that we found that this operation is commutative and associative, and that it distributes over vector addition $+$. We noted that $\mathbb{R}^2$ has a "unity element" $(1, 0)$ such that $(a, b) \times (1, 0) = (a, b)$ for all (a, b) and a "zero element" $(0, 0)$ such that $(a, b) + (0, 0) = (a, b)$. Putting all this together, we say that $\mathbb{R}^2$ has the structure of a **ring**. Finally, we noted that every nonzero element (a, b) has a multiplicative inverse, $(a, b)^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$. So $\mathbb{R}^2$ is more than just a ring; we call it a **field**.

Activity:

• Compute $(0, 1) \times (0, 1)$. What does this mean?

• If we think of $(1, 0)$ as "the number 1" and $(0, 1)$ as "the square root of -1", then we can think of a general vector (a, b) as

$$(a, b) = (a, 0) + (0, b) = a(1, 0) + b(0, 1) = a \cdot 1 + b \cdot \sqrt{-1}.$$

Show that this makes sense.

• Show that every vector (a, b) in $\mathbb{R}^2$ can be put in **polar form**, by writing $(a, b) = (r\cos\theta, r\sin\theta)$ for some real number $r \geq 0$ and angle θ.

• What happens if you multiply two vectors in polar form:

$$(r_1 \cos\theta_1, r_1 \sin\theta_1) \times (r_2 \cos\theta_2, r_2 \sin\theta_2) = \,?$$

• What happens if you raise a vector to the power of n:

$$(r\cos\theta, r\sin\theta)^n = \,?$$

• Use the result to prove **de Moivre's theorem**.

• Find all numbers z such that $z^3 + 1 = 0$. [Hint: Think of 1 as $(1, 0)$ and 0 as $(0, 0)$. Then put everything in polar form.]

• Find all numbers z such that $z^n + 1 = 0$.

Homework for Thursday November 13:

• Look up the Fundamental Theorem of Algebra.

• Look up Carl Friedrich Gauss. What does he have to do with the Fundamental Theorem of Algebra?

Day 20: more Funny Numbers

Homework:

• What did your learn about Carl Friedrich Gauss?

• What did you learn about the Fundamental Theorem of Algebra?

Recall:

• If we multiply vectors in polar form we get the following remarkable fact:

$$(r_1 \cos\theta_1, r_1 \sin\theta_1) \times (r_2 \cos\theta_2, r_2 \sin\theta_2) = (r_1 r_2 \cos(\theta_1 + \theta_2), r_1 r_2 \sin(\theta_1 + \theta_2)).$$

In other words, multiplying two vectors in $\mathbb{R}^2$ is the same as **multiplying the lengths** and **adding the angles**. (That's a lot more natural than the original definition we had in terms of rectilinear coordinates.) We proved this using the trigonometric "angle sum formulas":

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

But why are these formulas true?

Activity:

• Consider a vector (a, b) in $\mathbb{R}^2$. We can think of this vector as a **function** that sends vectors to vectors via the rule

$$(x, y) \mapsto (a, b) \times (x, y).$$

What does the function $(0, 1)$ do?

• Functions can be **composed**. What happens if we **first** apply the function (c,d) and **then** apply the function (a,b)? [Hint: The result is a function that corresponds to some other vector. Which other vector?]

• Show that the function $(\cos\alpha, \sin\alpha)$ **rotates** each vector counterclockwise by angle α. [Hint: Show that it rotates the vectors $(1,0)$ and $(0,1)$ each by angle α. Why does this imply that it rotates a general vector (x,y) by angle α?]

• Use the previous results to argue that for all angles α and β we have

$$(\cos\alpha, \sin\alpha) \times (\cos\beta, \sin\beta) = (\cos(\alpha+\beta), \sin(\alpha+\beta)).$$

[Hint: Don't do any calculations. Just think about what this function does.] This is the **correct** proof of the trigonometric angle sum identities.

• Define the function $f(x) = \cos x + \sin x \cdot \sqrt{-1}$ and show that $f(x)f(y) = f(x+y)$. What functions do you know that act like this? [Hint: You know all of them.]

• The previous problem suggests that we should write $f(x) = e^{rx}$ for some constant r. Which constant r should we choose? [Hint: Consider $f'(0)$.]

Homework for Thursday November 20:

• Choose a topic for Independent Study 2 and set up a meeting with me to discuss it.

Day 21: The Square Root of 2

Homework:

• Did you choose a topic for IS2? Did you set up a meeting with me?

Definition:

Today we will discuss the idea of **continued fractions**. Given any real number α we define a sequence of fractions $\alpha_0, \alpha_1, \alpha_2, \ldots$ and a sequence of integers $a_0, a_1, a_2, \ldots$ by the following algorithm. Start by defining $\alpha_0 := \alpha$. Then for all $n \geq 1$ we recursively define

- $a_n := \lfloor \alpha_n \rfloor$,
- $\alpha_{n+1} := \begin{cases} \frac{1}{\alpha_n - a_n} & \text{if } \alpha_n - a_n > 0 \\ \text{STOP} & \text{if } \alpha_n - a_n = 0 \end{cases}.$

In other words,

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

We use the more compact notation $\alpha = [a_0; a_1, a_2, a_3, \ldots]$ and we call this the **continued fraction expansion** of α.

Activity:

• Compute the continued fraction expansion of $\frac{12}{41}$.

• Compute the continued fraction expansion of $-\frac{12}{41}$.

• If α is a fraction, explain why the denominators of the sequence of fractions $\alpha_0, \alpha_1, \alpha_2, \ldots$ strictly decrease. In this case, explain why the continued fraction expansion must be **finite** (i.e., the algorithm terminates).

• Compute the continued fraction expansion of $\sqrt{2}$.

• Explain why $\sqrt{2}$ is not a fraction.

• Given a continued fraction expansion $\alpha = [a_0; a_1, a_2, a_3, \ldots]$, we say that the fractions represented by the prefixes $[a_0]$, $[a_0; a_1]$, $[a_0; a_1, a_2]$, etc., are the **convergents** of α. One can show that this is the sequence of "best possible" fractional approximations of α. Compute the first several convergents of $\sqrt{2}$.

• Use the same idea to come up with a very accurate fractional approximation of π.

Discussion:

• Are continued fractions preferable to the decimal system? Why or why not?

Homework for Tuesday December 2:

• Find sources for IS2 and start reading. Don't put it off until the last minute.

Day 22: more Square Root of 2

Homework:

• How is IS2 going? I am available to meet this week if you want to discuss your topic.

Recall:

Last time we discussed the continued fraction expansion of $\sqrt{2}$:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}$$

Since the expansion is infinite we know that $\sqrt{2}$ is not a fraction. The prefixes of the continued fraction give us the sequence of best rational approximations:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \ldots \longrightarrow \sqrt{2}.$$

Activity:

• Draw the curves $x^2 - 2y^2 = 1$ and $x^2 - 2y^2 = -1$ in the Cartesian plane.

• Let's think of the fraction $\frac{a}{b}$ (in lowest terms) as the point (a, b) in the Cartesian plane. Plot the first few rational approximations of $\sqrt{2}$ as points in the plane. What do you see? Can you prove that your observation is correct?

• Let $\frac{p}{q}$ be a "best rational approximation" to $\sqrt{2}$. Show that the next fraction in the sequence is $\frac{p+2q}{p+q}$. Show that the one after that is $\frac{3p+4q}{2p+3q}$.

• If (p, q) is a solution to the equation $x^2 - 2y^2 = \pm 1$, show that $(3p+4q, 2p+3q)$ is a solution to the same equation. Why does this prove that your observation above is correct?

• Something more is going on here. Given two vectors (a, b) and (c, d) in the plane $\mathbb{R}^2$, let's define another **strange multiplication**:

$$(a, b) * (c, d) := (ac + 2bd, ad + bc).$$

Show that $(3, 2)^n$ is a solution to the Diophantine equation $x^2 - 2y^2 = 1$ for all integers $n \geq 1$.

• How should we define $(3, 2)^0$? How should we define $(3, 2)^{-1}$?

• Hmm. That last one is tricky. The vector $(3, 2)^{-1}$ should satisfy the equation

$$(3, 2)^{-1} * (3, 2) = (1, 0).$$

How do we know that there is such a vector? [Hint: Think of (a, b) as the number $a + b\sqrt{2}$. To compute $(a, b)^{-1}$ we should "rationalize the denominator".] It is amazing fact that that the **complete solution** to $x^2 - 2y^2 = 1$ is $\pm(3, 2)^n$ for all integers n.

Homework for Thursday December 4:

• Work on IS2.

• Look up **Pell's Equation**. Who is it named after?

Day 23: more Square Root of 2

Homework:

• What did you learn about Pell's Equation?

Recall:

• Last time we discussed the Diophantine equations $x^2 - 2y^2 = 1$. We observed that the vector $\pm(3, 2)^n$ is a solution for any integer n, where the exponent is understood in terms of the strange multiplication

$$(a, b) * (c, d) := (ac + 2bd, ad + bc).$$

And we claimed without proof that this is the **complete solution**. But what **is** this funny product? Today we'll try to understand it by putting vectors in an appropriate "polar form".

Definition:

• By analogy with the usual trigonometric functions

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \text{and} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

we define the **hyperbolic functions**

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Activity:

- Show that $(\cosh x)^2 - (\sinh x)^2 = 1$.

- Show that the points $(\cosh x, \sinh x/\sqrt{2})$ live on the hyperbola $x^2 - 2y^2 = 1$.

- Show that (almost) any vector (a, b) can be written in the "polar form" $(r \cosh x, r \sinh x/\sqrt{2})$.

- What is the analogue of Euler's formula $e^{ix} = \cos x + i \sin x$ in the hyperbolic case?

- What happens if you strangely multiply two vectors in polar form:

$$(r_1 \cosh x_1, r_1 \sinh x_1/\sqrt{2}) * (r_2 \cosh x_2, r_2 \sinh x_2/\sqrt{2}) = \ ?$$

[Hint: You can save a lot of time by applying the hyperbolic "Euler's formula".] What does this remind you of?

- What kind of multiplication would you use to solve the Diophantine equation

$$x^2 - Dy^2 = 1 \ ?$$

Epilogue: If D is a positive non-square integer, then it is a theorem that the Diophantine equation $x^2 - Dy^2 = 1$ has complete solution $\pm(a, b)^n$ where (a, b) is some **fundamental solution** and the exponentiation is with regard to an appropriate multiplication. However, it might not be easy to find this fundamental solution. For example, when $D = 61$ the fundamental solution is $(1766319049, 226153980)$.

The problem of when $x^2 - Dy^2 = -1$ has a solution is still open.

Homework for the Last Day of Class, Tuesday December 9:

- Finish IS2.

Day 24: Review of What We Did

Homework:

- Hand in your IS2.

Review:

- Pythagorean Theorem:

$$c^2 = a^2 + b^2 \quad \Longleftrightarrow \quad \theta = 90^\circ$$

- Volume of a cone with base area B and perpendicular height h:

$$\frac{1}{3} Bh$$

- Volume of a sphere with radius r:

$$\frac{4}{3}\pi r^3$$

- Surface area of a sphere with radius r:

$$4\pi r^2$$

- Area of a triangle with angles α, β, γ on the surface of a sphere with radius r:

$$r^2(\alpha + \beta + \gamma - \pi)$$

- Distance between points (x_1, y_1) and (x_2, y_2):

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

- Equation of a circle with center (a, b) and radius r:

$$(x - a)^2 + (y - b)^2 = r^2$$

- Equation of a line perpendicular to vector (a, b):

$$ax + by + c = 0$$

- The lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are perpendicular when

$$aa' + bb' = 0$$

- Distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

- Equation of a sphere with center (a, b, c) and radius r:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

- Equality of vectors:

$$[\mathbf{x}, \mathbf{y}] = [\mathbf{u}, \mathbf{v}] \quad \Longleftrightarrow \quad \mathbf{y} - \mathbf{x} = \mathbf{v} - \mathbf{u}$$

- Addition of vectors:

$$[\mathbf{u}, \mathbf{v}] + [\mathbf{v}, \mathbf{w}] = [\mathbf{u}, \mathbf{w}]$$

- Subtraction of vectors:

$$[\mathbf{x}, \mathbf{y}] = [\mathbf{0}, \mathbf{y}] - [\mathbf{0}, \mathbf{x}]$$

- Abuse of notation:

$$[\mathbf{x}, \mathbf{y}] = \mathbf{y} - \mathbf{x}$$

- Distance between points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$:

$$\sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

- Dot product of vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$:

$$\mathbf{x} \bullet \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$$

- Length of a vector $\mathbf{x}$:

$$\|x\|^2 = \mathbf{x} \bullet \mathbf{x}$$

- Distance between points $\mathbf{x}$ and $\mathbf{y}$:

$$\|\mathbf{y} - \mathbf{x}\| = \sqrt{(\mathbf{y} - \mathbf{x}) \bullet (\mathbf{y} - \mathbf{x})}$$

- Distributive property of dot product:

$$\mathbf{x} \bullet (a\,\mathbf{y} + b\,\mathbf{z}) = a\,\mathbf{x} \bullet \mathbf{y} + b\,\mathbf{x} \bullet \mathbf{z}$$

- Consequence of the distributive property:

$$\|\mathbf{y} - \mathbf{x}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\mathbf{x} \bullet \mathbf{y}$$

- Law of Cosines:

$$c^2 = a^2 + b^2 - 2\,ab\cos\theta$$

- Law of Cosines:

$$\|\mathbf{y} - \mathbf{x}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$$

- Geometric interpretation of dot product:

$$\mathbf{x} \bullet \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$$

- Angle between vectors $\mathbf{x}$ and $\mathbf{y}$:

$$\arccos\left(\frac{\mathbf{x} \bullet \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right)$$

- Pythagorean Theorem:

$$\mathbf{x} \bullet \mathbf{y} = 0 \quad \Longleftrightarrow \quad \mathbf{x} \perp \mathbf{y}$$

- Equation of a plane perpendicular to vector (a, b, c):

$$ax + by + cz + d = 0$$

- Equation of a line in $\mathbb{R}^3$:

There is no such thing.

- Parametrization of a line in $\mathbb{R}^3$:

$$(x, y, z) = (x_0, y_0, z_0) + t(u, v, w)$$

- Parametrization of the unit circle $x^2 + y^2 = 1$:

$$(x, y) = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2}\right)$$

- Rational points on the unit circle $x^2 + y^2 = 1$:

$$(x, y) = \left(\frac{u^2 - v^2}{u^2 + v^2}, \frac{2uv}{u^2 + v^2}\right)$$

- Pythagorean Triples $a^2 + b^2 = c^2$:

$$(a, b, c) = d \cdot (u^2 - v^2, 2uv, u^2 + v^2)$$

- Parametrization of the circle $x^2 + y^2 = 2$:

$$(x, y) = \left(\frac{1 + 2t - t^2}{1 + t^2}, \frac{t^2 + 2t - 1}{1 + t^2} \right)$$

- Rational points on the circle $x^2 + y^2 = 2$:

$$(x, y) = \left(\frac{u^2 + 2uv - v^2}{u + 2 + v^2}, \frac{v^2 + 2uv - u^2}{u^2 + v^2} \right)$$

- Solution of the Diophantine equation $a^2 + b^2 = 2c^2$:

$$(a, b, c) = d \cdot (u^2 + 2uv - v^2, v^2 + 2uv - u^2, u^2 + v^2)$$

- Rational points on the circle $x^2 + y^2 = 3$:

No rational points exist.

- Fermat's Christmas Theorem:

Let d be an integer. Then the circle $x^2 + y^2 = d$ contains rational points if and only if the prime factors of d of the form $p = 3 \bmod 4$ occur with even multiplicity.

- Diophantus' Two-Square Identity:

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

- Funny multiplication on $\mathbb{R}^2$:

$$(a, b) \times (c, d) = (ac - bd, ad + bc)$$

- Interpretation of funny multiplication:

$$(a, b) = \text{“}a + b\sqrt{-1}\text{”}$$

- Theorem of funny numbers:

The set $\mathbb{R}^2$ together with vector addition $+$ and funny multiplication $\times$ forms a field with multiplicative identity $(1, 0)$ and additive identity $(0, 0)$.

- Interpretation of funny multiplication:

If we rotate the vector (x, y) counterclockwise by θ and scale it by r, we obtain the vector $(r \cos\theta, r \sin\theta) \times (x, y)$.

- De Moivre's Theorem:

$$(\cos\theta_1 + \sin\theta_1\sqrt{-1})(\cos\theta_2 + \sin\theta_2\sqrt{-1}) = \cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)\sqrt{-1}$$

- Euler's Formula:

$$e^{\theta\sqrt{-1}} = \cos\theta + \sin\theta\sqrt{-1}$$

- Interpretation of circular functions:

$$\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2} \qquad \text{and} \qquad \sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$$

- Pythagorean Theorem:

$$(\cos x)^2 + (\sin x)^2 = 1$$

- Continued fraction expansion of $\sqrt{2}$:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}$$

- Brahmagupta's Identity:

$$(a^2 - 2b^2)(c^2 - 2d^2) = (ac + 2bd)^2 + 2(ad + bc)^2$$

- Strange multiplication on $\mathbb{R}^2$:

$$(a, b) * (c, d) = (ac + 2bd, ad + bc)$$

- Interpretation of strange multiplication:

$$(a, b) = \text{“}a + b\sqrt{2}\text{”}$$

- Theorem of strange numbers:

The set $\mathbb{R}^2$ with vector addition $+$ and strange multiplication $*$ forms a field with multiplicative identity $(1, 0)$ and additive identity $(0, 0)$.

- Definition of hyperbolic functions:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

- Hyperbolic Pythagorean Theorem:

$$(\cosh x)^2 - (\sinh x)^2 = 1$$

- Hyperbolic Euler's Formula:

$$e^x = \cosh x + \sinh x$$

- Hyperbolic De Moivre's Theorem:

$$(\cosh t_1 + \sinh t_1)(\cosh t_2 + \sinh t_2) = \cosh(t_1 + t_2) + \sinh(t_1 + t_2)$$

- Parametrization of the hyperbola $x^2 - 2y^2 = 1$:

$$(x, y) = \pm(\cosh t, \sinh t/\sqrt{2})$$

- Integer points on the hyperbola $x^2 - 2y^2 = 1$:

$$(x, y) = \pm(3, 2) * (3, 2) * \cdots * (3, 2)$$

Homework:

- Meet at Chipotle (6290 S Dixie Hwy) tomorrow at noon.