Math 161 **Summer 2023**
Homework 4 Solutions **Drew Armstrong**

3.7.2. Find the most general antiderivative.

$$f(x) = 8x^9 - 3x^6 + 12x^3$$
$$\int f(x)\,dx = 8 \cdot \frac{x^{10}}{10} - 3 \cdot \frac{x^7}{7} + 12 \cdot \frac{x^4}{4} + C.$$

3.7.12. Find the most general antiderivative.

$$\begin{aligned}
f(x) &= 2\sqrt{x} + 6\cos x \\
f(x) &= 2x^{1/2} + 6\cos x \\
\int f(x)\,dx &= 2 \cdot \frac{x^{1/2+1}}{1/2+1} + 6\sin x + C \\
\int f(x)\,dx &= 2 \cdot \frac{x^{3/2}}{3/2} + 6\sin x + C \\
\int f(x)\,dx &= 2 \cdot \frac{2}{3}x^{3/2} + 6\sin x + C
\end{aligned}$$

3.7.18. Find f.

$$\begin{aligned}
f''(x) &= x^6 - 4x^4 + x + 1 \\
f'(x) = \int f''(x)\,dx &= \frac{x^7}{7} - 4 \cdot \frac{x^5}{5} + \frac{x^2}{2} + x + C_1 \\
f'(x) &= \frac{1}{7} \cdot x^7 - \frac{4}{5} \cdot x^5 + \frac{1}{2} \cdot x^2 + x + C_1 \\
f(x) = \int f'(x)\,dx &= \frac{1}{7} \cdot \frac{x^8}{8} - \frac{4}{5} \cdot \frac{x^6}{6} + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{x^2}{2} + C_1x + C_2 \\
f(x) &= \frac{1}{56} \cdot x^8 - \frac{4}{30} \cdot x^6 + \frac{1}{6} \cdot x^3 + \frac{1}{2} \cdot x^2 + C_1x + C_2.
\end{aligned}$$

3.7.22. Find f.

$$\begin{aligned}
f'''(t) &= t - \sqrt{t} \\
f'''(t) &= t - t^{1/2} \\
f''(t) = \int f'''(t)\,dt &= \frac{t^2}{2} - \frac{t^{1/2+1}}{1/2+1} + C_1 \\
f''(t) &= \frac{1}{2} \cdot t^2 - \frac{2}{3} \cdot t^{3/2} + C_1 \\
f'(t) = \int f''(t)\,dt &= \frac{1}{2} \cdot \frac{t^3}{3} - \frac{2}{3} \cdot \frac{t^{3/2+1}}{3/2+1} + C_1t + C_2 \\
f'(t) &= \frac{1}{6} \cdot t^3 - \frac{2}{3} \cdot \frac{2}{5} \cdot t^{5/2} + C_1t + C_2 \\
f'(t) &= \frac{1}{6} \cdot t^3 - \frac{4}{15} \cdot t^{5/2} + C_1t + C_2
\end{aligned}$$

$$\begin{aligned} f(t) = \int f'(t)\,dt &= \frac{1}{6}\cdot\frac{t^4}{4} - \frac{4}{15}\cdot\frac{t^{5/2+1}}{5/2+1} + C_1\cdot\frac{t^2}{2} + C_2t + C_3 \\ f(t) &= \frac{1}{24}\cdot t^4 - \frac{4}{15}\cdot\frac{2}{7}\cdot t^{7/2} + C_1\cdot\frac{t^2}{2} + C_2t + C_3 \\ f(t) &= \frac{1}{24}\cdot t^4 - \frac{8}{105}\cdot t^{7/2} + C_1\cdot\frac{t^2}{2} + C_2t + C_3. \end{aligned}$$

4.3.2. Evaluate the integral.

$$\begin{aligned} \int_1^2 \left(4x^3 - 3x^2 + 2x\right)\,dx &= \left[4\cdot\frac{x^4}{4} - 3\cdot\frac{x^3}{3} + 2\cdot\frac{x^2}{2}\right]_1^2 \\ &= \left[x^4 - x^3 + x^2\right]_1^2 \\ &= \left[2^4 - 2^3 + 2^2\right] - \left[1^4 - 1^3 + 1^2\right] \\ &= [16 - 8 + 4] - [1 - 1 + 1] \\ &= 12 - 1 \\ &= 11. \end{aligned}$$

4.3.2. (Wrong Version) Evaluate the integral.

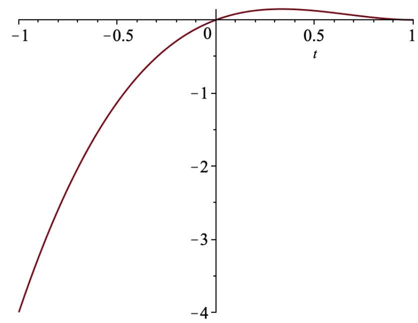
$$\begin{aligned} \int_1^2 \left(4x^3 - 3x^2 - 2x\right)\,dx &= \left[4\cdot\frac{x^4}{4} - 3\cdot\frac{x^3}{3} - 2\cdot\frac{x^2}{2}\right]_1^2 \\ &= \left[x^4 - x^3 - x^2\right]_1^2 \\ &= \left[2^4 - 2^3 - 2^2\right] - \left[1^4 - 1^3 - 1^2\right] \\ &= [16 - 8 - 4] - [1 - 1 - 1] \\ &= 4 - (-1) \\ &= 5. \end{aligned}$$

4.3.6. Evaluate the integral.

$$\begin{aligned} \int_{-1}^1 t(1-t)^2\,dt &= \int_{-1}^1 t(1 - 2t + t^2)\,dt \\ &= \int_{-1}^1 (t - 2t^2 + t^3)\,dt \\ &= \left[\frac{t^2}{2} - 2\cdot\frac{t^3}{3} + \frac{t^4}{4}\right]_{-1}^1 \\ &= \left[\frac{1}{2} - 2\cdot\frac{1}{3} + \frac{1}{4}\right] - \left[\frac{(-1)^2}{2} - 2\cdot\frac{(-1)^3}{3} + \frac{(-1)^4}{4}\right] \\ &= \left[\frac{1}{2} - 2\cdot\frac{1}{3} + \frac{1}{4}\right] - \left[\frac{1}{2} + 2\cdot\frac{1}{3} + \frac{1}{4}\right] \\ &= \frac{1}{12} - \frac{17}{12} \\ &= -\frac{16}{12} \end{aligned}$$

$$= -\frac{4}{3}.$$

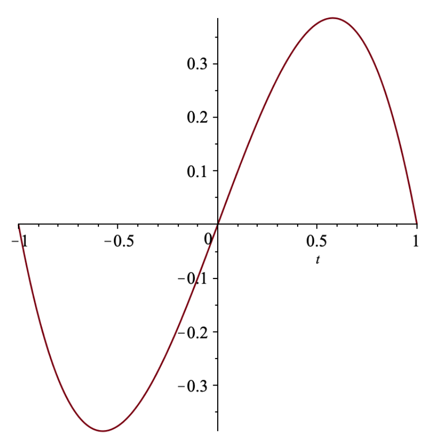
Remark: We got a negative number because there is more negative area than positive area in this region:



4.3.6. (Wrong Version) Evaluate the integral.

$$\begin{aligned}\int_{-1}^{1} t(1-t^2)\,dt &= \int_{-1}^{1} (t-t^3)\,dt \\ &= \left[\frac{t^2}{2}-\frac{t^4}{4}\right]_{-1}^{1} \\ &= \left[\frac{1^2}{2}-\frac{1^4}{4}\right]-\left[\frac{(-1)^2}{2}-\frac{(-1)^4}{4}\right] \\ &= \left[\frac{1}{2}-\frac{1}{4}\right]-\left[\frac{1}{2}-\frac{1}{4}\right] \\ &= 0.\end{aligned}$$

Remark: We got zero because positive and negative area cancel:

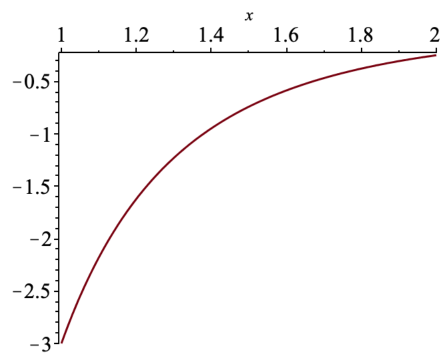


4.3.8. Evaluate the integral.

$$\int_{1}^{2} \left(\frac{1}{x^2}-\frac{4}{x^3}\right)\,dx = \int_{1}^{2} \left(x^{-2}-4\cdot x^{-3}\right)\,dx$$

$$
\begin{aligned}
&= \left[\frac{x^{-1}}{-1} - 4\cdot\frac{x^{-2}}{-2}\right]_1^2 \\
&= \left[-\frac{1}{x} + \frac{2}{x^2}\right]_1^2 \\
&= \left[-\frac{1}{2} + \frac{2}{2^2}\right] - \left[-\frac{1}{1} + \frac{2}{1^2}\right] \\
&= \left[-\frac{1}{2} + \frac{1}{2}\right] - [-1+2] \\
&= 0 - 1 \\
&= -1.
\end{aligned}
$$

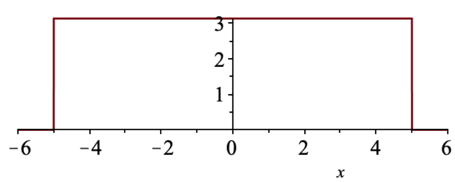
Remark: The area is negative because the graph is below the x-axis when $1 < x < 2$:



4.3.20. Evaluate the integral.

$$\int_{-5}^{5} \pi\, dx = [\pi x]_{-5}^{5} = [5\pi] - [-5\pi] = 10\pi.$$

Remark: This is just the area of area of a rectangle with width 10 and height π:



4.3.24. Evaluate the integral. Here the textbook expects us to remember that $(\sec\theta)' = \sec\theta\tan\theta$. Then we get

$$
\begin{aligned}
\int_{\pi/4}^{\pi/3} \sec\theta\tan\theta\, d\theta &= [\sec\theta]_{\pi/4}^{\pi/3} \\
&= \sec(\pi/3) - \sec(\pi/4) \\
&= \frac{1}{\cos(\pi/3)} - \frac{1}{\cos(\pi/3)} \\
&= \frac{1}{1/2} - \frac{1}{1/\sqrt{2}} \\
&= 2 - \sqrt{2}.
\end{aligned}
$$

Remark: If you don't remember that $(\sec\theta)' = \sec\theta\tan\theta$ we can use substitution. First we simplify:

$$\begin{aligned}\int \sec\theta\tan\theta\,d\theta &= \int \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}\,d\theta\\ &= \int \frac{\sin\theta}{(\cos\theta)^2}\,d\theta.\end{aligned}$$

Then we let $u = \cos\theta$ so that $du/d\theta = -\sin\theta$ and $d\theta = -du/\sin\theta$. Hence

$$\begin{aligned}\int \sec\theta\tan\theta\,d\theta &= \int \frac{\sin\theta}{(\cos\theta)^2}\,d\theta\\ &= \int \frac{\cancel{\sin\theta}}{u^2}\cdot\frac{du}{-\cancel{\sin\theta}}\\ &= -\int \frac{1}{u^2}\,du\\ &= -\int u^{-2}\,du\\ &= -\frac{u^{-2+1}}{-2+1}+C\\ &= -\frac{u^{-1}}{-1}+C\\ &= \frac{1}{u}+C\\ &= \frac{1}{\cos\theta}+C\\ &= \sec\theta + C.\end{aligned}$$

4.4.6. Use the Fundamental Theorem of Calculus to compute the derivative.

$$\begin{aligned}g(x) &= \int_1^x (2+t^4)^5\,dt\\ g'(x) &= (2+x^4)^5.\end{aligned}$$

4.4.8. Use the Fundamental Theorem of Calculus to compute the derivative.

$$\begin{aligned}F(x) &= \int_x^{10} \tan\theta\,d\theta\\ F(x) &= -\int_{10}^x \tan\theta\,d\theta\\ F'(x) &= -\tan x.\end{aligned}$$

4.4.10. Use the Fundamental Theorem of Calculus to compute the derivative.

$$h(x) = \int_0^{x^2} \sqrt{1+r^3}\,dr$$

First we consider the nicer function

$$f(x) = \int_0^x \sqrt{1+r^3}\,dr,$$

which satisfies $f'(x) = \sqrt{1+x^3}$ by the Fundamental Theorem. Then we use the chain rule:

$$\begin{aligned} h(x) &= f(x^2) \\ h'(x) &= f'(x^2) \cdot (x^2)' \\ h'(x) &= f'(x^2) \cdot (2x) \\ h'(x) &= \sqrt{1+(x^2)^3} \cdot 2x. \end{aligned}$$

4.5.2. Evaluate the integral using the substitution $u = 2 + x^4$. First we note that

$$\begin{aligned} du/dx &= 4x^3 \\ du &= 4x^3 dx \\ dx &= du/(4x^3). \end{aligned}$$

Then we have

$$\begin{aligned} \int x^3(2+x^4)^5\,dx &= \int x^3u^5\,dx \\ &= \int \cancel{x^3}u^5\left(\frac{du}{4\cancel{x^3}}\right) \\ &= \frac{1}{4}\int u^5\,du \\ &= \frac{1}{4}\cdot\frac{u^6}{6} + C \\ &= \frac{1}{24}(2+x^4)^6 + C. \end{aligned}$$

4.5.4. Evaluate the integral using the substitution $u = 1 - 6t$. First we note that

$$\begin{aligned} du/dt &= -6 \\ du &= -6dt \\ dt &= -du/6. \end{aligned}$$

Then we have

$$\begin{aligned} \int \frac{dt}{(1-6t)^4}\,dt &= \int \frac{-du/6}{u^4} \\ &= -\frac{1}{6}\int u^{-4}\,du \\ &= -\frac{1}{6}\frac{u^{-4+1}}{-4+1} + C \\ &= -\frac{1}{6}\cdot\frac{1}{-3}\cdot\frac{1}{u^3} + C \\ &= \frac{1}{18}\cdot\frac{1}{(1-6t)^3} + C. \end{aligned}$$

4.5.8. Evaluate the indefinite integral.

$$\int x^2\cos(x^3)\,dx$$

We will use the substitution $u = x^3$ so that

$$\begin{aligned} du/dx &= 3x^2 \\ du &= 3x^2 dx \\ dx &= du/(3x^2). \end{aligned}$$

Then we have

$$\begin{aligned} \int x^2 \cos(x^3)\, dx &= \int \cancel{x^2} \cos(u) \cdot \frac{du}{3\cancel{x^2}} \\ &= \frac{1}{3} \int \cos u \, du \\ &= \frac{1}{3} \cdot \sin u + C \\ &= \frac{1}{3} \cdot \sin(x^3) + C. \end{aligned}$$

4.5.10. Evaluate the indefinite integral.

$$\int (3t+2)^{2.4}\, dt$$

We will use the substitution $u = 3t + 2$ so that

$$\begin{aligned} du/dt &= 3 \\ du &= 3dt \\ dt &= du/3. \end{aligned}$$

Then we have

$$\begin{aligned} \int (3t+2)^{2.4}\, dt &= \int u^{2.4} \cdot \frac{du}{3} \\ &= \frac{1}{3} \int u^{2.4}\, du \\ &= 3 \cdot \frac{u^{3.4}}{3.4} + C \\ &= \frac{1}{3(3.4)} \cdot (3t+2)^{3.4} + C. \end{aligned}$$

4.5.14. Evaluate the indefinite integral.

$$\int \frac{x}{(x^2+1)^2}\, dx$$

We will use the substitution $u = x^2 + 1$ so that

$$\begin{aligned} du/dx &= 2x \\ du &= 2xdx \\ dx &= du/(2x). \end{aligned}$$

Then we have

$$\begin{aligned} \int \frac{x}{(x^2+1)^2}\, dx &= \int \frac{\cancel{x}}{u^2} \cdot \frac{du}{2\cancel{x}} \\ &= \frac{1}{2} \int u^{-2}\, du \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \cdot \frac{u^{-2+1}}{-2+1} + C \\
&= -\frac{1}{2}\frac{1}{u} + C \\
&= -\frac{1}{2(x^2+1)} + C
\end{aligned}
$$

4.5.20. Evaluate the integral.

$$\int \cos^4\theta \sin\theta \, d\theta$$

We will use the substitution $u = \cos\theta$ so that

$$
\begin{aligned}
du/d\theta &= -\sin\theta \\
du &= -\sin\theta d\theta \\
d\theta &= -du/\sin\theta.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\int \cos^4\theta \sin\theta \, d\theta &= \int u^4 \cancel{\sin\theta} \cdot \frac{du}{-\cancel{\sin\theta}} \\
&= -\int u^4 \, du \\
&= -\frac{u^5}{5} + C \\
&= -\frac{\cos^5\theta}{5} + C.
\end{aligned}
$$

A.1. Since r is constant we have

$$
\begin{aligned}
\int_{-r}^{r} \pi(r^2 - x^2)\, dx &= \pi \int_{-r}^{r} (r^2 - x^2)\, dx \\
&= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} \\
&= \pi \left(\left[r^2 \cdot r - \frac{r^3}{3} \right] - \left[r^2(-r) - \frac{(-r)^3}{3} \right] \right) \\
&= \pi \left(\left[r^3 - \frac{r^3}{3} \right] - \left[-r^3 + \frac{r^3}{3} \right] \right) \\
&= \pi \left(\frac{2}{3} r^3 - \left[-\frac{2}{3} r^3 \right] \right) \\
&= \frac{4}{3} \pi r^3,
\end{aligned}
$$

which is the correct formula for the volume of a sphere.