**Math 161** **Summer 2023**

**Exam 2 Solutions** **Drew Armstrong**

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**1A. Use the limit definition** to compute the derivative of $f(x) = 5x^2$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{5(x+h)^2 - 5x^2}{h} \\ &= \lim_{h\to 0} \frac{5(\cancel{x^2} + 2xh + h^2) - \cancel{5x^2}}{h} \\ &= \lim_{h\to 0} \frac{5\cancel{h}(2x+h)}{\cancel{h}} \\ &= 5(2x+0) \\ &= 10x. \end{aligned}$$

**1B. Use the limit definition** to compute the derivative of $f(x) = x^2 + 1$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[(x+h)^2 + \cancel{1}] - [x^2 + \cancel{1}]}{h} \\ &= \lim_{h\to 0} \frac{(\cancel{x^2} + 2xh + h^2) - \cancel{x^2}}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} \\ &= 2x+0 \\ &= 2x. \end{aligned}$$

**2A.** Compute the derivative of $f(x) = (\sin x)^2$. We use the power rule and chain rule:

$$f'(x) = 2(\sin x)^1 \cdot (\sin x)' = 2\sin x \cos x.$$

**2B.** Compute the derivative of $f(x) = (\cos x)^2$. We use the power rule and chain rule:

$$f'(x) = 2(\cos x)^1 \cdot (\cos x)' = 2(\cos x)(-\sin x) = -2\sin x \cos x.$$

**3A.** Compute the derivative of $f(x) = \dfrac{1+x}{1+x^2}$. We use the quotient rule:

$$f'(x) = \frac{(1+x^2)(1+x)' - (1+x)(1+x^2)'}{(1+x^2)^2} = \frac{(1+x^2)(0+1) - (1+x)(0+2x)}{(1+x^2)^2}.$$

**3B.** Compute the derivative of $f(x) = \dfrac{1+x^2}{1+x}$. We use the quotient rule:

$$f'(x) = \frac{(1+x)(1+x^2)' - (1+x^2)(1+x)'}{(1+x)^2} = \frac{(1+x^2)(0+2x) - (1+x)(0+1)}{(1+x)^2}.$$

**4A.** Compute the derivative of $f(x) = x \cdot \sin(\sqrt{x})$. We use the product and chain rules:

$$\begin{aligned} f'(x) &= (x)' \cdot \sin(\sqrt{x}) + x \cdot (\sin(\sqrt{x}))' \\ &= (1) \cdot \sin(\sqrt{x}) + x \cdot \cos(\sqrt{x}) \cdot (\sqrt{x})' \\ &= \sin(\sqrt{x}) + x \cdot \cos(\sqrt{x}) \cdot \left(\frac{1}{2\sqrt{x}}\right). \end{aligned}$$

**4B.** Compute the derivative of $f(x) = \cos x \cdot \sin(x^2)$. We use the product and chain rules:

$$\begin{aligned} f'(x) &= (\cos x)' \cdot \sin(x^2) + \cos x \cdot (\sin(x^2))' \\ &= (-\sin x) \cdot \sin(x^2) + \cos x \cdot \cos(x^2) \cdot (x^2)' \\ &= (-\sin x) \cdot \sin(x^2) + \cos x \cdot \cos(x^2) \cdot (2x). \end{aligned}$$

**5A.** Find the equation of the tangent line to the curve $\frac{x^2}{8} + \frac{y^2}{18} = 1$ at the point $(x, y) = (2, 3)$. The slope of the tangent line is $dy/dx$ which we find using implicit differentiation:

$$\begin{aligned} \left(\frac{x^2}{8} + \frac{y^2}{18}\right)' &= (1)' \\ \frac{1}{8}(x^2)' + \frac{1}{18}(y^2)' &= 0 \\ \frac{1}{8}(2x) + \frac{1}{18}(2yy') &= 0 \\ \frac{1}{9}yy' &= -\frac{1}{4}x \\ y' &= -\frac{9x}{4y}. \end{aligned}$$

At the point $(x, y) = (2, 3)$ we have $y' = -(9 \cdot 2)/(4 \cdot 3) = -3/2$, so the equation of the tangent line is $-3/2 = (y - 3)/(x - 2)$.

**5B.** Find the equation of the tangent line to the curve $\frac{x^2}{2} + \frac{y^2}{8} = 1$ at the point $(x, y) = (1, 2)$. The slope of the tangent line is $dy/dx$ which we find using implicit differentiation:

$$\begin{aligned} \left(\frac{x^2}{2} + \frac{y^2}{8}\right)' &= (1)' \\ \frac{1}{2}(x^2)' + \frac{1}{8}(y^2)' &= 0 \\ \frac{1}{2}(2x) + \frac{1}{8}(2yy') &= 0 \\ \frac{1}{4}yy' &= -x \\ y' &= -\frac{4x}{y}. \end{aligned}$$

At the point $(x, y) = (1, 2)$ we have $y' = -(4 \cdot 1)/2 = -2$, so the equation of the tangent line is $-2 = (y - 2)/(x - 1)$.