7/31/15

HW5 due now.
Quiz 5 on Monday.

Discuss HW5 solutions

Summary of material for Quiz 5:

- Chap 4.5: Substitution
- Chap 5.1: Inverse Functions
- Chap 5.2 – 5.4: Log & Exp.
- Chap 5.6: Inverse Trig Functions
- Chap 6.1: Integration by Parts

(We skipped 5.7 & 5.8).

So what's next?

On Tues & Wed next week we will review for the Final Exam.

Today & Monday I'll show you some cool applications of Calculus.

Yesterday I mentioned the "normal distribution"

$$N(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Actually, this is just the "standard" normal distribution. For any numbers μ and $\sigma > 0$ we define

$$N(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

This is called the "normal distribution with mean μ and standard deviation σ". Let's sketch the graph:

Since μ & σ are constant, we have

$$N'(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-(x-\mu)^2/(2\sigma^2)} \cdot \left(\frac{-(x-\mu)^2}{2\sigma^2} \right)'$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \cdot \frac{-1}{\cancel{2}\sigma^3} \cdot \cancel{2}(x-\mu)^1 \cdot 1$$

$$= \frac{-(x-\mu)}{\sigma^3\sqrt{2\pi}} \cdot e^{-(x-\mu)^2/(2\sigma^2)}$$

Since $e^t > 0$ for all t we find that

- $N'(x; \mu, \delta) = 0$ when $x = \mu$
- $N'(x; \mu, \delta) > 0$ when $x < 0$
- $N'(x; \mu, \delta) < 0$ when $x > 0$.

Is $x = \mu$ a max or min?

Let's compute $N''(x; \mu, \delta)$. Just kidding. I'll spare you the gory details and just tell you that

$$N''(x; \mu, \delta) = \frac{(x^2 - 2\mu x + \mu^2 - \delta^2)}{2\delta^5 \sqrt{\pi}} \cdot e^{-(x-\mu)^2/(2\delta^2)}.$$

Since $e^t > 0$ for all t we just need to look at the quadratic equation

$$x^2 - 2\mu x + (\mu^2 - \delta^2) = 0.$$

$$x = \frac{2\mu \pm \sqrt{4\cancel{\mu^2} - 4(\cancel{\mu^2} - \delta^2)}}{2}$$

$$= \frac{2\mu \pm \sqrt{4\sigma^2}}{2}$$

$$= \frac{2\mu \pm 2\sigma}{2}$$

$$= \mu \pm \sigma .$$

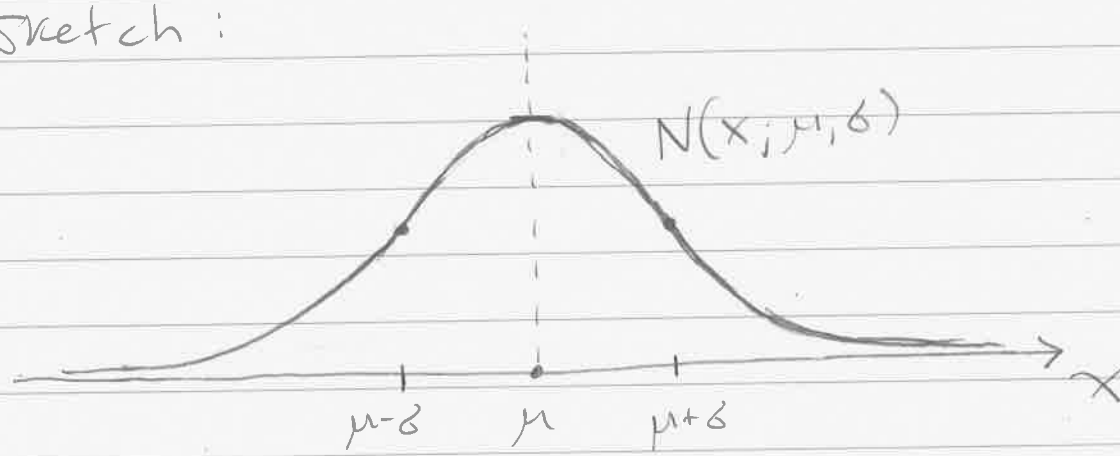
We have inflection points at

$$x = \mu \pm \sigma .$$

Also we have $N''(\mu; \mu, \sigma) = \dfrac{-1}{\sigma^3\sqrt{2\pi}} < 0$

so $x = \mu$ is a local <u>maximum</u>.

Sketch:

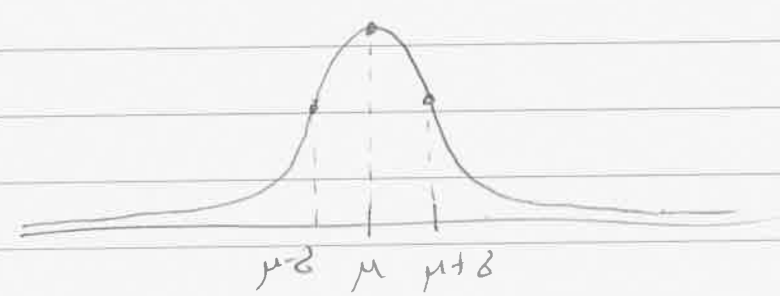


Why do we care about this function?
There is a fancy theorem in probability called the Central Limit Theorem that says the following:

"If you repeat an experiment many times then the distribution of outcomes will look like $N(x; \mu, \sigma)$ for some μ & σ."

For example, I am currently performing this experiment: Teach Calculus to 19 students. Assign them HW, Quizzes and a Final Exam. Grade the assignments and give each student a score out of 100.

The C.L.T. tells us that the distribution should look like



[Except 19 is not a big number, so it probably won't look like this.]

Problem: Suppose the average score is $\mu = 75$ and the standard deviation is $\sigma = 10$. How many students had scores between 70 and 80?

We want to compute the area under the curve $N(x; 75, 10)$ from $x = 70$ to $x = 80$

$$\int_{70}^{80} N(x; 75, 10)\, dx = ?$$

☆ Sad Fact: It is impossible to write down the exact value of this integral in terms of any functions we know.

So our only options are to use a computer or to look it up in a table. I'll show you how to look it up in a table.

First we'll use a substitution to express everything in terms of the standard normal distribution.

Fact: Using the substitution $x = z\sigma + \mu$ gives us

$$N(x;\mu,\sigma) = N(z;0,1)/\sigma$$

[You can check this but it's tedious.]

Then since $dx = \sigma \cdot dz$ we have

$$\int_{x_1}^{x_2} N(x;\mu,\sigma)\,dx = \int_{\frac{x_1-\mu}{\sigma}}^{\frac{x_2-\mu}{\sigma}} \frac{N(z;0,1)}{\cancel{\sigma}} \cdot \cancel{\sigma}\,dz$$

In our case,

$$\mu = 75, \quad \sigma = 10, \quad x_1 = 70, \quad x_2 = 8,$$

$$\frac{x_1-\mu}{\sigma} = -\frac{1}{2}, \quad \frac{x_2-\mu}{\sigma} = +\frac{1}{2}.$$

So we get

$$\int_{70}^{80} N(x;75,80)\,dx = \int_{-1/2}^{1/2} N(z;0,1)\,dz$$

OK, now what?

Well, people need to integrate the standard normal distribution so often that you can find a table of values in the back of any statistics book.

Define $\Phi(z) := \int_{-\infty}^{z} N(z;0,1)\,dz$

so that $\Phi'(z) = N(z;0,1)$.

By the Fundamental Theorem of Calculus we have

$$\int_{-1/2}^{1/2} N(z;0,1)\,dz = \Phi\left(\tfrac{1}{2}\right) - \Phi\left(-\tfrac{1}{2}\right)$$

Now Look it up!

The back of the book says

$$\Phi(+\tfrac{1}{2}) \approx 0.6914$$

$$\Phi(-\tfrac{1}{2}) \approx 0.3085$$

So $\Phi(+\tfrac{1}{2}) - \Phi(-\tfrac{1}{2}) \approx 0.3829$

We conclude that

$$\int_{70}^{80} N(x; 75, 10)\, dx \approx 0.3829$$

What does this mean? The area under the curve should be interpreted as probability. It means that approximately 38.3% of the students scored between 70 and 80. Since there are 19 students in the class this is

$$19(0.3829) = 7.28 \text{ students.}$$

[This is why I always tell you the mean μ and standard deviation σ after each HW and Quiz.]

8/3/15

Quiz 5 now (25 minutes).

HW5	Total	30
	Ave.	26.1
	Med	28.0
	St. Dev.	4.0

Tomorrow & Wed we will review for the Final Exam which is on Friday, here at 10:05 – 12:05.

Today I'll show you a bit more of how Calculus is relevant to your life.

Recall from Problem A1 on HW2 the following limit: If r is constant then we have

$$\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n = e^r .$$

Now I'll explain what this means.

Problem: Suppose you invest \$1000 and it grows with approximately 6% annual rate of return. How much will you have after 5 years?

Instead of solving this specific problem let's first investigate the general situation:

You invest P dollars at and it grows with annual rate of return $r > 0$. How much will you have after t years?

It depends if the bank reports r as APR (annual percentage rate) or as APY (annual percentage yield). The APR is reported <u>without</u> compound interest and the APY is reported <u>with</u> compound interest.

Let's assume that r is the APR. How much you actually get depends on how often the interest is calculated (i.e. compounded).

Let's assume that interest is compounded n times per year. This means that after every $(1/n)$-th of a year, the bank multiplies your money by $\left(1+\frac{r}{n}\right)$. So the amount of money looks like

$$P \cdot \left(1+\frac{r}{n}\right)\left(1+\frac{r}{n}\right)\left(1+\frac{r}{n}\right) \cdots$$

If you wait for t years, how many times will the interest be compounded? Answer: Exactly $n \cdot t$ times. So after t years, you will have

$$P \cdot \underbrace{\left(1+\frac{r}{n}\right)\left(1+\frac{r}{n}\right) \cdots \left(1+\frac{r}{n}\right)}_{n \cdot t \text{ times}}$$

$$= P\left(1+\frac{r}{n}\right)^{nt} \text{ dollars}$$

Let's assume your bank is generous and compounds interest <u>daily</u> so that $n = 365$. Then after t years you will have

$$P\left(1+\frac{r}{365}\right)^{365t} \text{ dollars.}$$

But this is hard to calculate so let's make the simplifying assumption that n is very large. Then we want to compute the limit as $n \to \infty$.

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = ?$$

[In this case we say that the interest is continuously compounded.]

Let's make the substitution $n = mr$. Then since $r > 0$ we have $m \to \infty$ as $n \to \infty$. We find that

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = \lim_{m\to\infty} P\left(1+\frac{1}{m}\right)^{mrt}$$

$$= P \cdot \lim_{m\to\infty} \left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}$$

$$= P \cdot \left[\lim_{m\to\infty} \left(1+\frac{1}{m}\right)^{m}\right]^{rt}$$

$$= P \cdot e^{rt}.$$

///

In summary: If you invest P dollars at annual percentage rate (APR) $r > 0$, then after t years you will have

$$P \cdot e^{rt} \text{ dollars.}$$

[Remark: The APY reports the actual percent that your money grows in one year, including compounding. If the APR is r and interest is continuously compounded, then the APY is

$$e^{r} - 1 .$$
]

Back to the specific problem. We have $P = \$1000$, $r = 6\%$ APR and $t = 5$ years. The amount of money is

$$1000 \cdot e^{(0.06)\cdot 5} = \$1349.86 .$$

The APY is

$$e^{0.06} - 1 = 0.0618$$
$$= 6.18\%$$

Now let's think of money as a function of time:

$$f(t) = P \cdot e^{rt}$$

Let's compute the derivative:

$$f'(t) = (P \cdot e^{rt})'$$

$$= P(e^{rt})'$$

$$= Pe^{rt} \cdot (rt)'$$

$$= Pe^{rt} \cdot r$$

$$= r \cdot f(t).$$

That's interesting. What does it mean? It means that at any moment, the rate of growth of the money equals r times the amount you currently have. I guess that make sense:

"The more you have, the more you get."

This is a very general phenomenon. It applies to:

- compound interest
- growth of a population under ideal conditions.
- radioactive decay
- heat transfer between two bodies.

In each of these situations the relevant function $f(t)$ satisfies the differential equation

$$\boxed{f'(t) = r \cdot f(t)} \qquad (*)$$

for some constant r.

If we didn't already know the solution, how could we solve equation $(*)$ to find $f(t)$?

First divide both sides by $f(t)$:

$$\frac{f'(t)}{f(t)} = r.$$

Now compute the antiderivative of each side with respect to t:

$$\int \frac{f'(t)}{f(t)}\,dt = \int r\,dt$$

Since r is constant, the right hand side evaluates to

$$\int r\,dt = rt + C$$

for some constant C. To compute the left hand side let $u = f(t)$ so that $du = f'(t)\,dt$. Then we have

$$\int \frac{f'(t)}{f(t)}\,dt = \int \frac{1}{u}\,du$$

$$= \ln|u| + D$$

$$= \ln\big(f(t)\big) + D$$

for some constant D. [We will assume that $f(t) > 0$ so that $|f(t)| = f(t)$.].

Now equate both sides to get

$$\ln(f(t)) + D = rt + C$$

$$\ln(f(t)) = rt + E$$

where E is some constant. Finally, take the exponential of both sides:

$$\exp(\ln(f(t))) = \exp(rt + E).$$

$$f(t) = e^{(rt+E)}$$

$$f(t) = e^{rt} \cdot e^{E}$$

$$\boxed{f(t) = K \cdot e^{rt}}$$

where K is some constant.

In summary, if $f'(t) = r \cdot f(t)$ for some constant r then $f(t) = K \cdot e^{rt}$ for some constant K.

Congratulations! We just solved the most important differential equation in the world.

Example: Let $P(t)$ be the population density at time t of some bacteria in a petri dish. If a given bacterium has enough space, food, water, etc. then it will reproduce at rate $r > 0$.

Thus, under ideal conditions the population will grow according to

$$P'(t) = r \cdot P(t)$$

"exponential growth".

This might be approximately true for small t, but we know this growth is not sustainable because the petri dish is finite. Suppose the petri dish has a certain fixed carrying capacity K.

Then as $P(t) \to K$ the rate of growth goes to zero, $P'(t) \to 0$. The easiest way to model this is via the "Logistic equation

$$P'(t) = r \cdot P(t) \left(K - P(t) \right).$$

Can we solve this differential equation? Yes.

First we write

$$\frac{dP}{dt} = r \cdot P(K-P)$$

$$\frac{dP}{P(K-P)} = r\,dt$$

Then we integrate both sides:

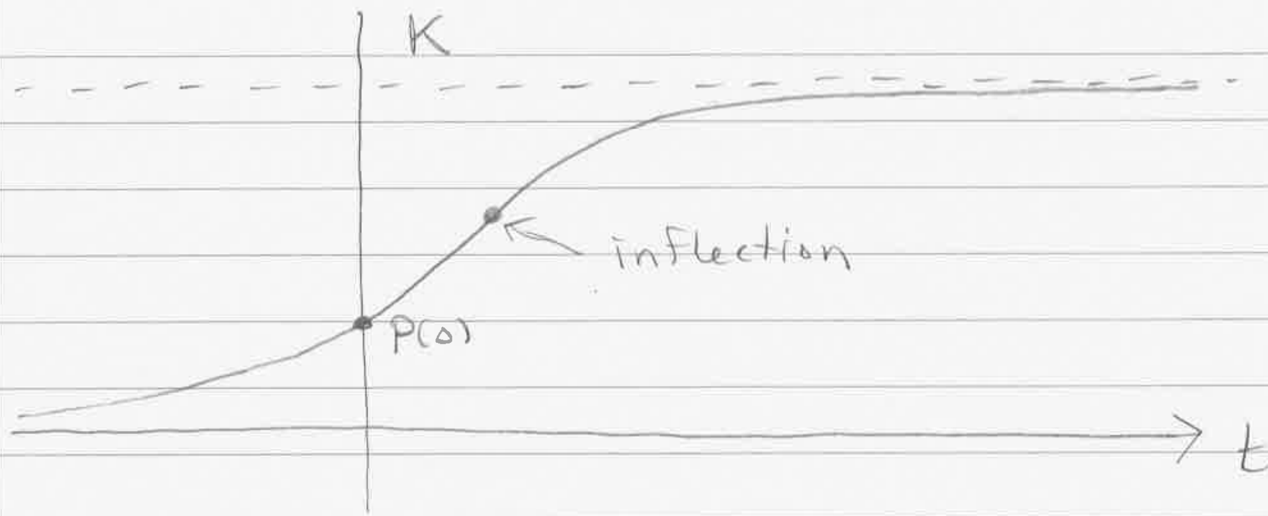
$$\int \frac{dP}{P(K-P)} = \int r\,dt$$

After a couple pages of work (omitted) we will find that

$$\boxed{P(t) = \frac{P(0)\cdot K}{P(0) + (K - P(0))\,e^{-Krt}}}$$

"logistic growth".

The graph of $P(t)$ looks like this:



This is called the "S-shaped curve" or the the "logistic curve". We hope that the human population $P(t)$ will behave like this in the future, and not in some scarier way.

8/4/15

Quiz 5	Total	10
	Ave	5.5
	Med	6.0
	St. Dev.	2.6

The Final Exam is on Friday in this room from 10:05 to 12:05.

It is 4.8 times longer than a Quiz. Since each Quiz had 5 problems, I estimate that the Final will have

$$4.8\,(5) = 24 \text{ problems}.$$

Today & tomorrow we will review the course material in preparation for the exam.

After an introduction, the course began with the concepts of

functions & limits.

Example: Consider the function

$$A(n) = \frac{n(n+1)(2n+1)}{6n^3}$$

If n is an integer then $f(n)$ is the area of n rectangles that approximate the area under the graph of $f(x) = x^2$ from $x = 0$ to $x = 1$. The exact area is

$$\int_0^1 x^2\,dx = \lim_{n \to \infty} A(n).$$

So we want to compute this limit. How?

After expanding the numerator we get

$$A(n) = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

So that

$$\lim_{n \to \infty} A(n) = \lim_{n \to \infty} \frac{1}{3} + \cancel{\lim_{n \to \infty} \frac{1}{2n}} + \cancel{\lim_{n \to \infty} \frac{1}{6n^2}}$$

$$= \frac{1}{3} + 0 + 0 = \frac{1}{3}.$$

Problem: Compute the limit

$$\lim_{n\to\infty} \frac{(2n-1)(3n-1)}{3n^2}.$$

Solution: $(2n-1)(3n-1) = 6n^2 - 5n + 1.$

$$\lim_{n\to\infty} \frac{(2n-1)(3n-1)}{3n^2} = \lim_{n\to\infty} \frac{6n^2-5n+1}{3n^2}$$

$$= \lim_{n\to\infty} \left[2 - \frac{5}{3n} + \frac{1}{3n^2} \right]$$

$$= 2 - 0 + 0 = 2.$$

Now let $f(x)$ be a function of a real variable x. We say that

$$\lim_{x\to a} f(x) = L$$

if the value of $f(x)$ approaches L as x approaches a. We don't care what happens when $x = a$; maybe $f(a)$ doesn't even exist!

Example: Compute $\lim_{x \to 2} \dfrac{x^2-4}{x-2}$.

If we define $f(x) = (x^2-4)/(x-2)$ then $f(2) =$ "$0/0$", which makes no sense (we call it an indeterminate form). So we will need some trick to evaluate the limit. In this case we can factor the numerator:

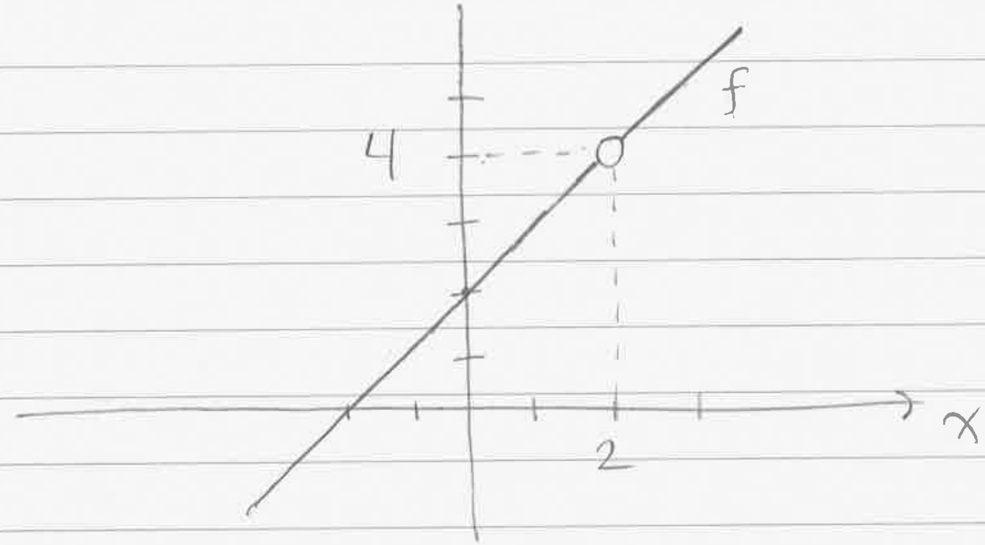
$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$$

$$= \lim_{x \to 2} (x+2) = 4.$$

We can say that

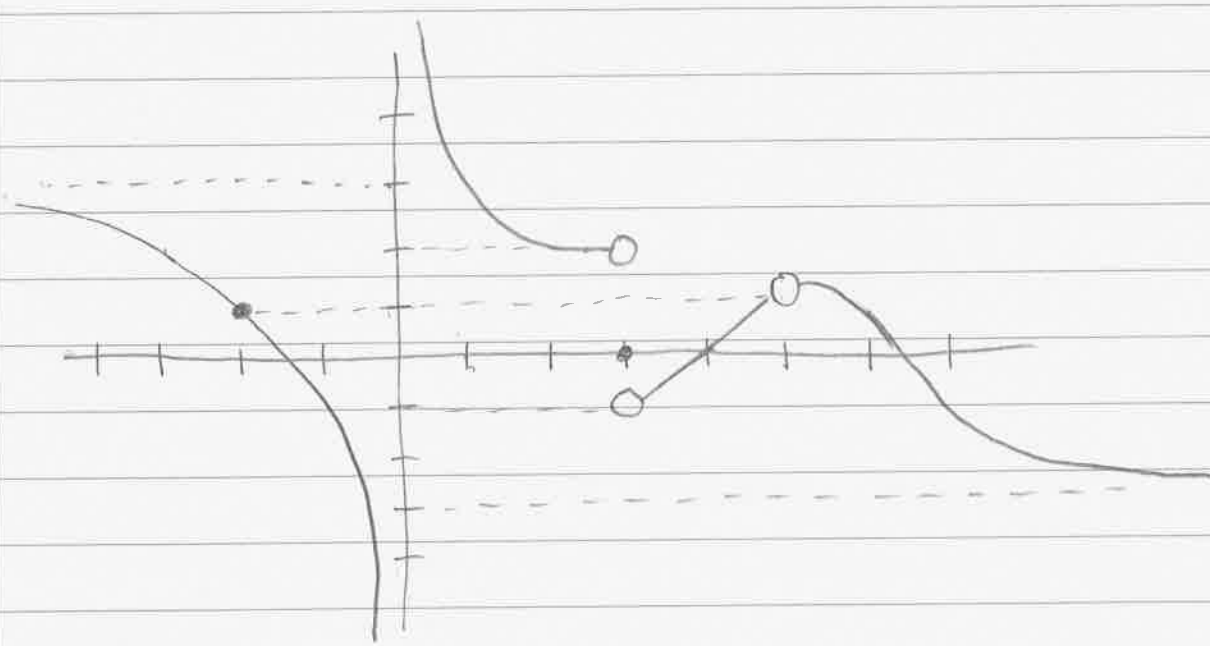
$$f(x) = \frac{x^2-4}{x-2} = \begin{cases} x+2 & \text{if } x \neq 2 \\ \text{undefined} & \text{if } x = 2. \end{cases}$$

The graph of this function looks like this:



The open circle indicates that $f(2)$ is not defined. And we can see from the graph that $\lim_{x \to 2} f(x) = 4$.

Example: Consider the following graph of the function $g(x)$:



Evaluate the following limits or say why they do not exist:

- $\lim_{x \to \infty} g(x) = -3$.

- $\lim_{x \to -\infty} g(x) = +3$

- $\lim_{x \to 0} g(x)$ does not exist.

  But can use the convenient notation

  " $\lim_{x \to 0^-} g(x) = -\infty$ " & " $\lim_{x \to 0^+} g(x) = +\infty$ "

  if we want.

- $\lim_{x \to -2} g(x) = 1$

- $\lim_{x \to 5} g(x) = 1$

  And it doesn't matter that $f(5)$ is undefined because the limit doesn't care.

- $\lim_{x \to 3} g(x)$ does not exist.

To be more specific we can say that

$\lim_{x \to 3^-} g(x) = 2$, $g(3) = 0$, $\lim_{x \to 3^+} g(x) = -1$.

Sometimes we need algebraic tricks to evaluate limits.

Example: Compute the limit.

$$\lim_{x \to 2} \frac{\sqrt{x+2} - 2}{x - 2} = ? \qquad \text{“}\frac{0}{0}\text{”}$$

Multiply top and bottom by the conjugate expression $\sqrt{x+2} + 2$ to get

$$\lim_{x \to 2} \frac{(\sqrt{x+2} - 2)}{(x-2)} \cdot \frac{(\sqrt{x+2} + 2)}{(\sqrt{x+2} + 2)}$$

$$= \lim_{x \to 2} \frac{(x+2) - 4}{(x-2)(\sqrt{x+2} + 2)}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{x+2}+2)}$$

$$= \lim_{x \to 2} \frac{1}{\sqrt{x+2}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}.$$

[Practice Problems : Chap 1.4 Ex. 11-28.].

The most important kind of limit is called a derivative.

Given a function $f(x)$ of a real variable x, we define its derivative by

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}}.$$

There are two ways to think of this:

1. If $f(t)$ is a function of time, then $f'(t)$ is the instantaneous rate of change of f at time t.

Example: If $s(t)$ is the height of a falling apple at time t then $s'(t)$ is the velocity and $s''(t)$ is the acceleration at time t.

2. $f'(a)$ is the slope of the tangent line to the graph of the function $f(x)$ at the point $(a, f(a))$.

Example: Find the equation of the tangent line to $f(x) = \sqrt{x}$ at $x = 4$.

The slope is

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h}$$

[Wait, we just did this. Make the substitution $h = u-2$, so $u \to 2$ as $h \to 0$.]

$$= \lim_{u \to 2} \frac{\sqrt{u+2} - 2}{u-2} = \frac{1}{4}.$$

//

The line of slope $f'(4) = 1/4$ containing the point $(4, f(4)) = (4, \sqrt{4}) = (4, 2)$ has the equation

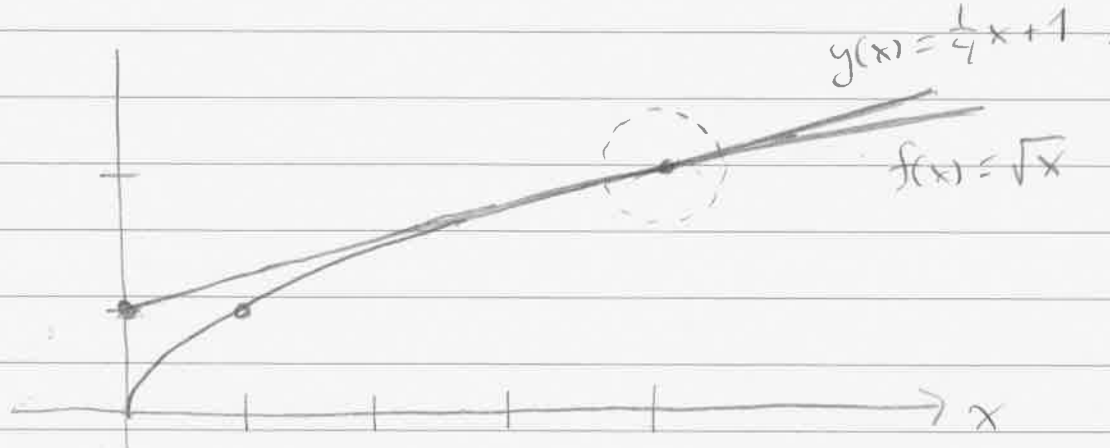
$$\frac{y-2}{x-4} = \frac{1}{4}$$

$$y - 2 = \frac{1}{4}(x-4)$$

$$y = \frac{1}{4}x - 1 + 2$$

$$\boxed{y = \frac{1}{4}x + 1}$$

Picture:



For $x \approx 4$ we have $\sqrt{x} \approx \frac{1}{4}x + 1$.

While were at it, let's use this information to approximate the value of $\sqrt{4.1}$.

Since $4.1 \approx 4$ we can use the linear approximation

$$\sqrt{4.1} \approx \frac{1}{4}(4.1) + 1$$

$$= 1.025 + 1 = 2.025.$$

For comparison, the true value is

$$\sqrt{4.1} = 2.02485\ldots$$

In general, given a function $f(x)$ we have

$$\boxed{f(x) \approx f(a) + f'(a)(x-a) \quad \text{when } x \approx a}$$

It is tedious to compute derivatives from the definition, so we are lucky to have many tricks and shortcuts.

Here is a summary:

- $(c)' = 0$ when c is constant
- $(x^p)' = p \cdot x^{p-1}$ when p is constant
- $(c \cdot f(x))' = c \cdot f'(x)$ when c is constant
- $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- $(\sin(x))' = \cos(x)$
- $(\cos(x))' = -\sin(x)$
- $(\log_b(x))' = \dfrac{1}{x \cdot \ln(b)}$ (b constant)
- $(b^x)' = \ln(b) \cdot b^x$ (b constant)
- Product Rule:

$$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x).$$

- Chain Rule:

$$(f(g(x)))' = f'(g(x)) \cdot g'(x).$$

Example: Use the product and chain rules to compute $(\tan(x))'$.

First write $\tan(x) = \dfrac{\sin(x)}{\cos(x)} = \sin(x)\cdot\cos(x)^{-1}$

Then we have

$$(\tan(x))' = \left(\sin(x)\cdot\cos(x)^{-1}\right)'$$

$$= (\sin(x))'\cdot\cos(x)^{-1} + \sin(x)\cdot\left(\cos(x)^{-1}\right)'$$

$$= \cancel{\cos(x)\cdot\cos(x)^{-1}} + \sin(x)\cdot(-1)\cos(x)^{-2}\cdot(\cos(x))'$$

$$= 1 - \frac{\sin(x)}{\cos^2(x)}\cdot(-\sin(x))$$

$$= 1 + \frac{\sin^2(x)}{\cos^2(x)}.$$

This can be simplified in various ways.

E.g. $1 + \dfrac{\sin^2(x)}{\cos^2(x)} = \dfrac{\cos^2(x)}{\cos^2(x)} + \dfrac{\sin^2(x)}{\cos^2(x)}$

$$= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}.$$

//

It is convenient to summarize this method as a rule:

- Quotient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

Proof: Using the product & chain rules gives

$$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x)g(x)^{-1}\right)'$$

$$= f'(x)\cdot g(x)^{-1} + f(x)\left(g(x)^{-1}\right)'$$

$$= f'(x)g(x)^{-1} + f(x)(-1)g(x)^{-2}g'(x)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2}$$

$$= \frac{f'(x)g(x)}{g(x)^2} - \frac{f(x)g'(x)}{g(x)^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

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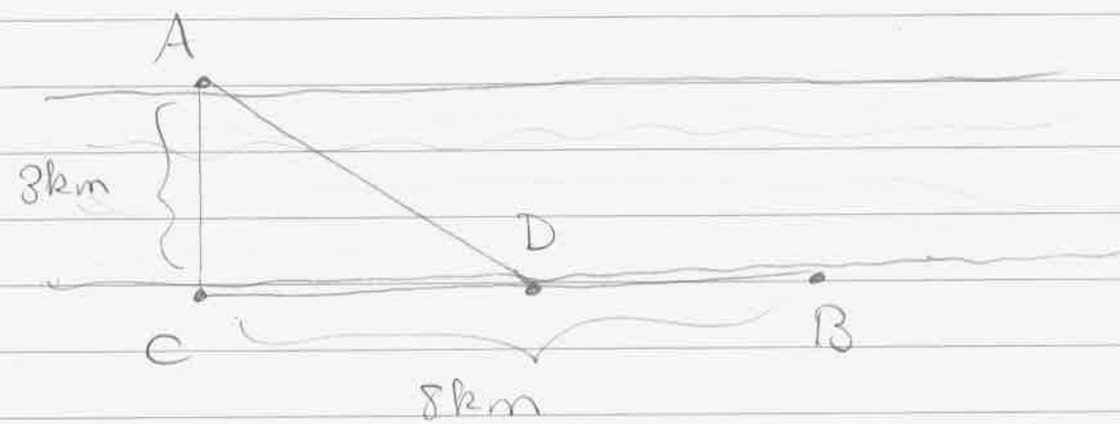
Application of Derivatives: You want to cross a 3km river from A to B.



You can row 6 km/h and you can run 8 km/h. Suppose you row from A to D and then run from D to B.

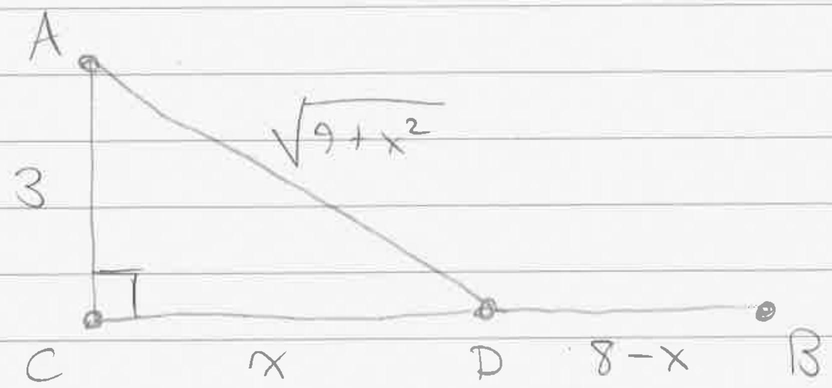
Where should D be to get from A to B as quickly as possible?

- First let's name things.

Let x be the distance from C to D and let T be the total travel time.

We want to minimize T as a function of x.

- To do this we need a formula for $T(x)$.



The distance from A to D is $\sqrt{9+x^2}$ km by the Pythagorean Theorem. If we row from A to D at 6 km/h this will take

$$\frac{\sqrt{9+x^2}}{6} \text{ hours.}$$

Then we run the $8-x$ km from D to B at 8 km/h, which takes

$$\frac{8-x}{8} \text{ hours.}$$

Therefore, the total travel time is

$$\boxed{T(x) = \frac{\sqrt{9+x^2}}{6} + \frac{8-x}{8}}$$

- Finally we apply Calculus.

$$T'(x) = \frac{1}{6}\cdot\frac{1}{2}(9+x^2)^{-\frac{1}{2}}(2x) + \frac{1}{8}(-1)$$

$$= \frac{x}{6\sqrt{9+x^2}} - \frac{1}{8}.$$

Setting $T'(x) = 0$ gives

$$\frac{x}{6\sqrt{9+x^2}} - \frac{1}{8} = 0$$

$$\frac{x}{6\sqrt{9+x^2}} = \frac{1}{8}$$

$$\frac{4x}{3} = \sqrt{9+x^2}$$

[Since $\sqrt{9+x^2} > 0$ this means that x must be positive.]

Square both sides to get

$$\frac{16x^2}{9} = 9 + x^2.$$

$$\frac{16}{9}x^2 - 1x^2 = 9$$

$$\frac{7}{9}x^2 = 9$$

$$x^2 = \frac{81}{7}$$

$$x = \frac{9}{+\sqrt{7}} \approx 3.4 \text{ km}.$$

[We choose the positive square root because x is positive.]

We check that $T'(x) < 0$ when $x < 9/\sqrt{7}$ and $T'(x) > 0$ when $x > 9/\sqrt{7}$ so that $T(x)$ has a MINIMUM when $x = 9/\sqrt{7} \approx 3.4$ km.

- We don't need to, but we could compute

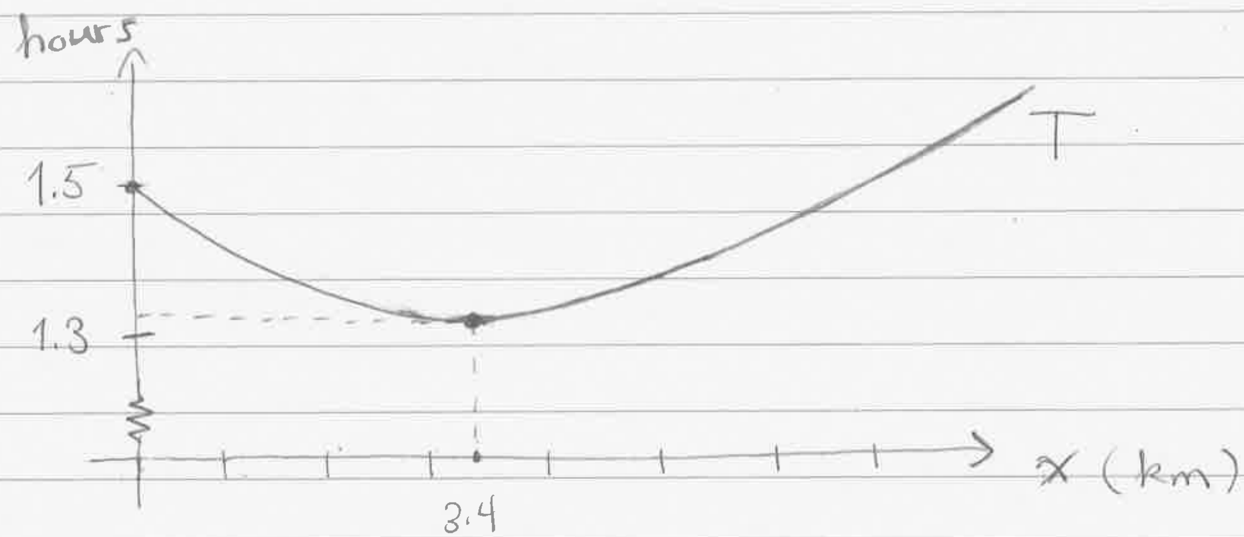
$$T''(x) = \frac{3}{2(9+x^2)^{3/2}}$$

Since $T''(x) > 0$ for all x we conclude that the graph of $T(x)$ is always concave up.

The minimum value of $T(x)$ is

$$T(3.4) = 1.33 \text{ hours}.$$

Finally, here is the graph:



8/5/15

Final Exam Friday
here from 10:05 – 12:05

You may use scientific calulators but not graphing calculators or phones, etc.

You may bring in one $8\frac{1}{2} \times 11$ sheet of paper with formulas (or whatever) on it.

Today: Review continued ...

Yesterday we reviewed functions, limits, and derivatives. We discussed some applications of derivatives. Here is one more.

Curve Sketching.

Let $f(x)$ be a function. Recall that

- $f(x)$ is increasing when $f'(x) > 0$
- $f(x)$ is decreasing when $f'(x) < 0$
- $f(x)$ is concave up when $f''(x) > 0$
- $f(x)$ is concave down when $f''(x) < 0$.
- If $f''(a) = 0$ we say that $f(x)$ has an inflection when $x = a$.

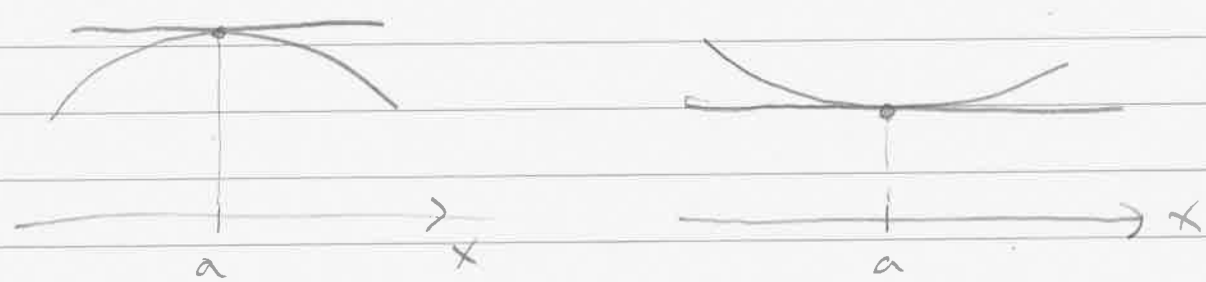
- If $f'(a) = 0$ then the slope of the tangent to the graph of $f(x)$ at $x = a$ is zero.

This could indicate that $x = a$ is a local maximum or minimum of f.

To distinguish these we can compute the second derivative.

☆ Second Derivative Test:

$f'(a) = 0$ & $f''(a) < 0 \Longrightarrow$ Local maximum
$f'(a) = 0$ & $f''(a) > 0 \Longrightarrow$ Local minimum
$f'(a) = 0$ & $f''(a) = 0 \Longrightarrow$ we don't know.



$f'(a) = 0$ & $f''(a) < 0$ $\qquad$ $f'(a) = 0$ & $f''(a) > 0$

Example: We are given

$$f(x) = 1 / \sqrt{3+x^2}$$

$$f'(x) = -x / (3+x^2)^{3/2}$$

$$f''(x) = (2x^2-3) / (3+x^2)^{5/2}.$$

Use this to sketch the graph of $f(x)$.

- Since $\sqrt{3+x^2} > 0$ for all x, we have

  $f(x) > 0$ for all x.

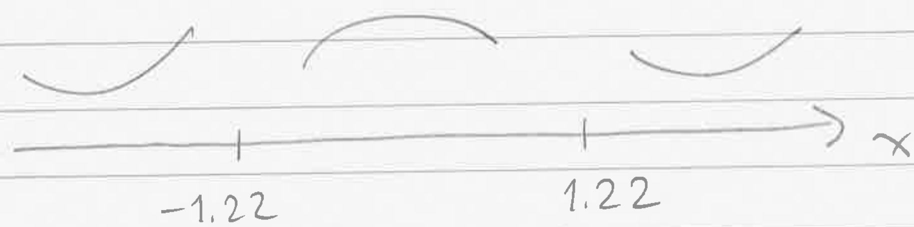
- Since $(3+x^2)^{3/2} > 0$ for all x, we have

  $f'(x) = 0$ when $x = 0$
  $f'(x) > 0$ when $x < 0$
  $f'(x) < 0$ when $x > 0$.

- Since $(3+x^2)^{5/2} > 0$ for all x, we have

$f''(x) = 0$ when

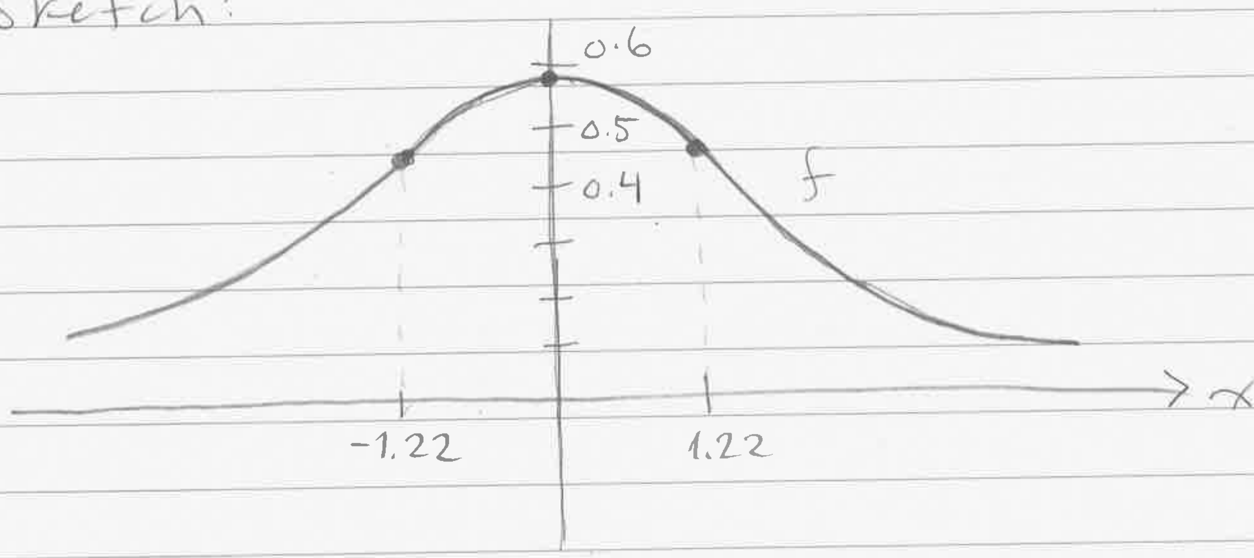
$$\begin{aligned} 2x^2 - 3 &= 0 \\ 2x^2 &= 3 \\ x^2 &= 3/2 \\ x &= \pm\sqrt{3/2}. \\ &\approx \pm 1.22 \end{aligned}$$



- There is a local max at $(0, f(0))$ $= (0, 1/\sqrt{3}) = (0, 0.577)$.

There are inflection points at $\left(-\sqrt{\frac{3}{2}}, f\left(-\sqrt{\frac{3}{2}}\right)\right)$ $= (-1.22, 0.47)$ and $\left(\sqrt{\frac{3}{2}}, f\left(\sqrt{\frac{3}{2}}\right)\right) = (1.22, 0.47)$.

Sketch:



Integration:

Let $f(x)$ be a function of a real variable x. The integral of $f(x)$ from $x=a$ to $x=b$ is defined as follows.

Given a positive integer n we let

$$\Delta x = (b-a)/n$$

$$x_i = a + i \cdot \Delta x$$

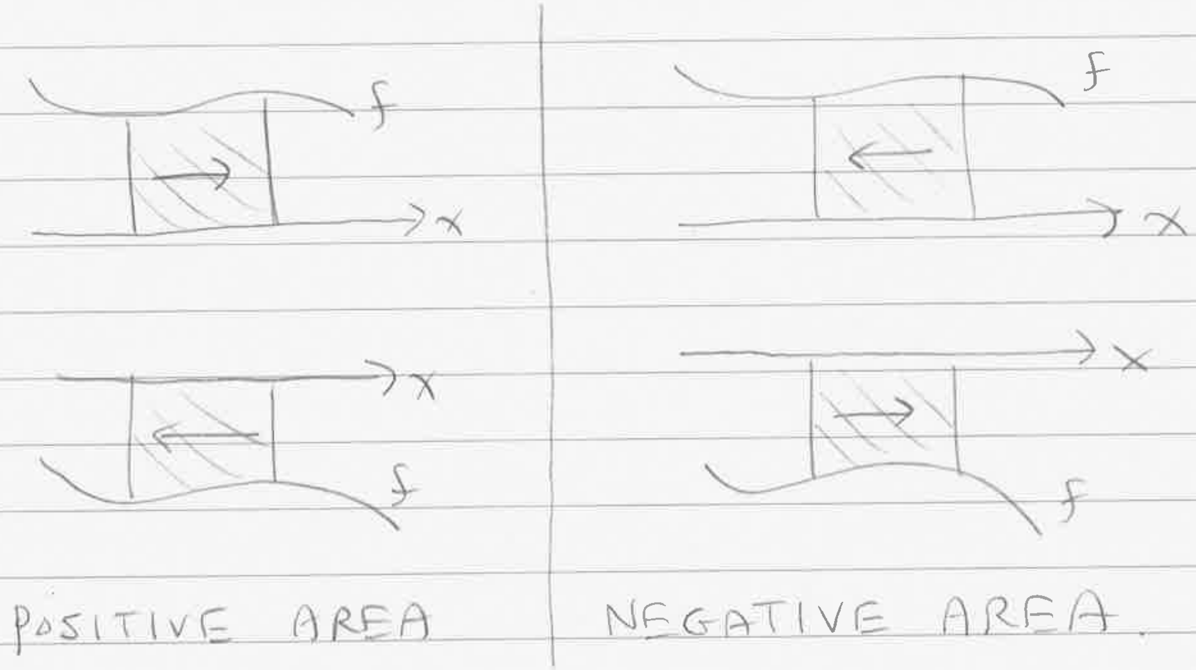
Then the integral is

$$\boxed{\int_a^b f(x)\,dx := \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\cdot\Delta x\right]}$$

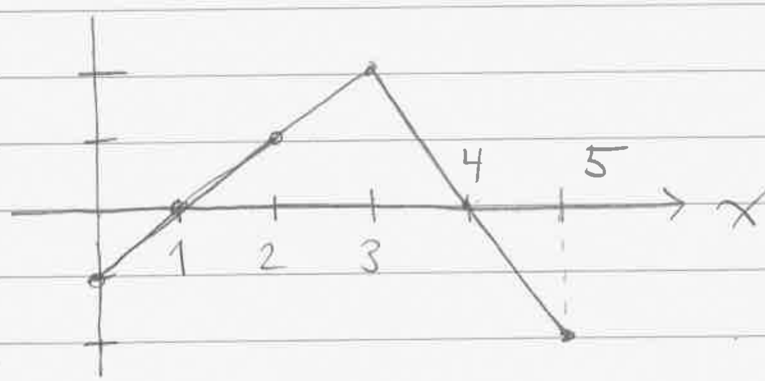
We interpret this as the "signed area" between the graph of $f(x)$ and the x-axis from $x=a$ to $x=b$.

Question: What does "signed area" mean?

Answer:



Example: Consider the graph of some g(x).



- $\int_0^1 g(x)\,dx = -\frac{1\cdot 1}{2} = -\frac{1}{2}$

- $$\int_1^4 g(x)\,dx = \frac{3\cdot 2}{2} = 3$$

- $$\int_4^5 g(x)\,dx = -\frac{1\cdot 2}{2} = -1$$

- $$\int_0^5 g(x)\,dx = -\frac{1}{2} + 3 - 1 = \frac{3}{2}.$$

- $$\int_3^5 g(x)\,dx = 1 - 1 = 0$$

- $$\int_3^0 g(x)\,dx = -\int_0^3 g(x)\,dx$$
$$= -\left(-\frac{1}{2} + 3\right) = -\frac{5}{2}.$$

General Formula: For all a, b, c we have

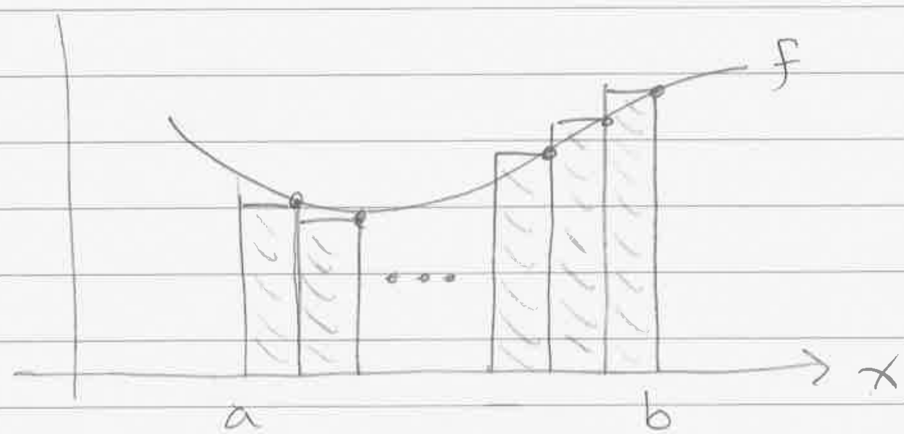
$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

The summation

$$R_n = \sum_{i=1}^{n} f\left(a + i\frac{(b-a)}{n}\right) \cdot \frac{(b-a)}{n}$$

is an approximation to the "signed area" under the curve using n skinny rectangles with heights defined by right hand endpoints.

Picture:



In practice, the limit

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} R_n$$

may be very difficult or impossible to evaluate. So we have a clever trick.

The Fundamental Theorem of Calculus:

Part 1: $$\boxed{\frac{d}{db}\int_a^b f(x)\,dx = f(b)}$$

This is the most important formula in Calculus because it gives us a practical method to compute integrals. In particular it tells us that an integral is basically an antiderivative.

If $F(x)$ is any function such that $F'(x) = f(x)$ then we have

Part 2: $$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Based on this theorem, we use the notation

$$\text{“} \int f(x)\,dx \text{”}$$

to denote the general antiderivative of $f(x)$.

The F.T.C. works because it's much easier to compute antiderivatives than it is to compute the limits $\lim_{n \to \infty} R_n$.

We have lots of special tricks:

- $\int k \, dx = kx + C$ for k constant
- $\int k \cdot f(x) \, dx = k \int f(x) \, dx$ for k constant
- $\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$
- $\int \cos x \, dx = \sin x + C$
- $\int \sin x \, dx = -\cos x + C$.
- $\int b^x \, dx = \frac{1}{\ln(b)} \cdot b^x + C$ (b constant).
- $\int x^p \, dx = \begin{cases} \frac{x^{p+1}}{p+1} & \text{if } p \neq -1 \\ \ln(x) & \text{if } p = -1 \end{cases}$

And some general-purpose tricks:

- Integration by Parts

$$\int f(x)\, g'(x)\, dx = f(x)g(x) - \int f(x) g'(x)\, dx$$

- Method of Substitution

$$\int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C.$$

But sometimes our tricks fail us. (antidifferentiation is harder than differentiation :( ).

Practice: Compute $\int_1^e \ln(x)\, dx$ and draw a picture of the region with this area.

First we'll compute an antiderivative of $\ln(x)$. We let $\ln(x) = f(x)\, g'(x)$ with $f(x) = \ln(x)$ and $g'(x) = 1$. Then we use integration by parts

$$\int f(x)\, g'(x)\, dx = f(x)g(x) - \int f'(x) g(x)\, dx$$

Since $f'(x) = 1/x$ and $g(x) = x$ we get

$$\int \ln(x)\,dx = x\ln(x) - \int \frac{1}{x}\cdot x\,dx$$

$$= x\ln(x) - \int 1\,dx$$

$$= x\ln(x) - x + C$$

We only need one particular antiderivative so we might as well take $C = 0$. Then we have

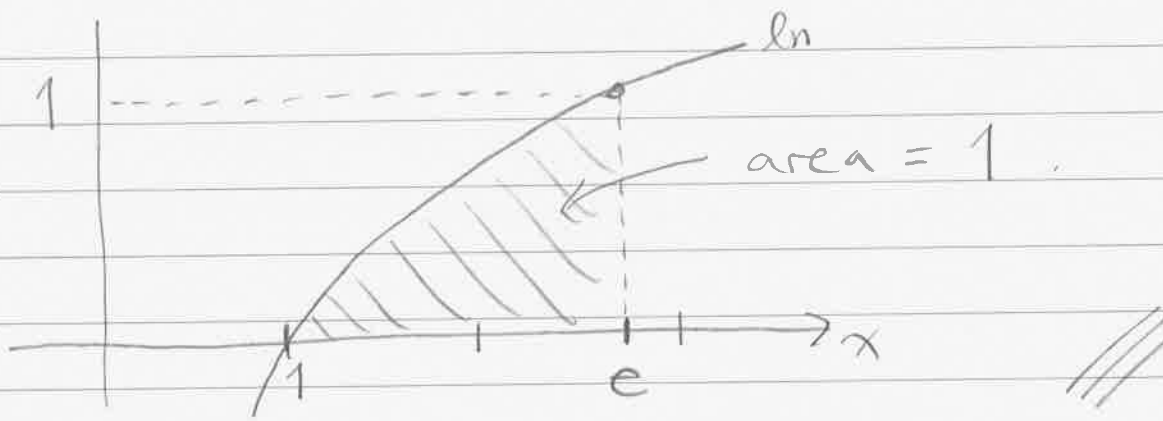
$$\int_1^e \ln(x)\,dx = \Big[\, x\ln(x) - x \,\Big]_1^e$$

$$= \Big[\big(e\underbrace{\ln(e)}_{1} - e\big) - \big(1\cdot\underbrace{\ln(1)}_{0} - 1\big)\Big]$$

$$= \big[(e-e) - (0-1)\big]$$

$$= 1.$$

Picture:



Integration is fresh in our minds so we won't dwell on it too much now.

Here's an issue I want to remind you of from Quiz 5 Problem 5.

Let f & g be inverse functions, so that

$$\boxed{y = f(x) \iff x = g(y)}$$

There is a formula relating the derivatives of f & g as follows.

Apply $\frac{d}{dx}$ to both sides of $x = g(y)$:

$$\frac{d}{dx} x = \frac{d}{dx} g(y)$$

$$1 = g'(y) \cdot \frac{dy}{dx}$$

Then solve for $f'(x) = \frac{dy}{dx}$ to get

$$f'(x) = \frac{dy}{dx} = \frac{1}{g'(y)} = \frac{1}{g'(f(x))}.$$

Similarly we have

$$g'(x) = \frac{1}{f'(g(x))}.$$

Example: Given the fact that $(\tan x)' = \frac{1}{\cos^2 x}$, compute $(\arctan x)'$.

Let $f(x) = \arctan x$ and $g(x) = \tan x$. Since these are inverse functions, we have

$$(\arctan x)' = f'(x)$$

$$= \frac{1}{g'(f(x))}$$

$$= \frac{1}{1/\cos^2(\arctan x)}$$

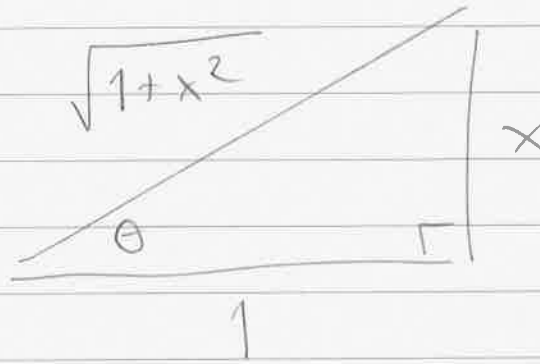
$$= \cos^2(\arctan x).$$

Can we simplify this? Yes. Let

$$\theta = \arctan x$$

so that $\tan\theta = x = x/1$.

Then we draw a right triangle with angle θ, "opposite" side of length x, and "adjacent" side of length 1:



The "hypotenuse" has length $\sqrt{1+x^2}$ by the Pythagorean theorem, so

$$\cos\theta = \frac{1}{\sqrt{1+x^2}}$$

i.e., $$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}.$$

Finally, we have

$$(\arctan x)' = \cos^2(\arctan x)$$
$$= \frac{1}{1+x^2}.$$