7/24/15

HW4 due now
Quiz 4 on Monday

First let's talk about the HW4 solutions and Quiz 4.

OK, we did that. What comes next?

We have now discussed all the main ideas of Calculus.

- Functions & Limits
- Derivatives
- Integrals
- The Fundamental Theorem.

But we need more practice, and there are some important kinds of functions and antiderivatives that we still don't know.

Chapter 5 will expand our catalog of functions to include logarithms, exponentials, and inverse trig functions.

Chapter 6 will give us new tricks for computing antiderivatives (and hence integrals).

Chapter 7 discusses some further applications of integrals (to computing volume, surface area, and arc length). Unfortunately we don't have time for this but I might show you something cool on the last day.

Let's start Chapter 5 by remembering the antiderivatives we know:

- $\int x^p \, dx = \frac{1}{p+1} \cdot x^{p+1}$ when $p \neq -1$.

- $\int \sin x \, dx = -\cos x$

- $\int \cos x \, dx = \sin x$

- $\int \sec^2 x \, dx = \tan x$.

That's pretty much it. There are still lots of functions we don't know how to antidifferentiate.

Maybe the most basic one is $x^{-1} = \frac{1}{x}$.

The power rule doesn't apply because

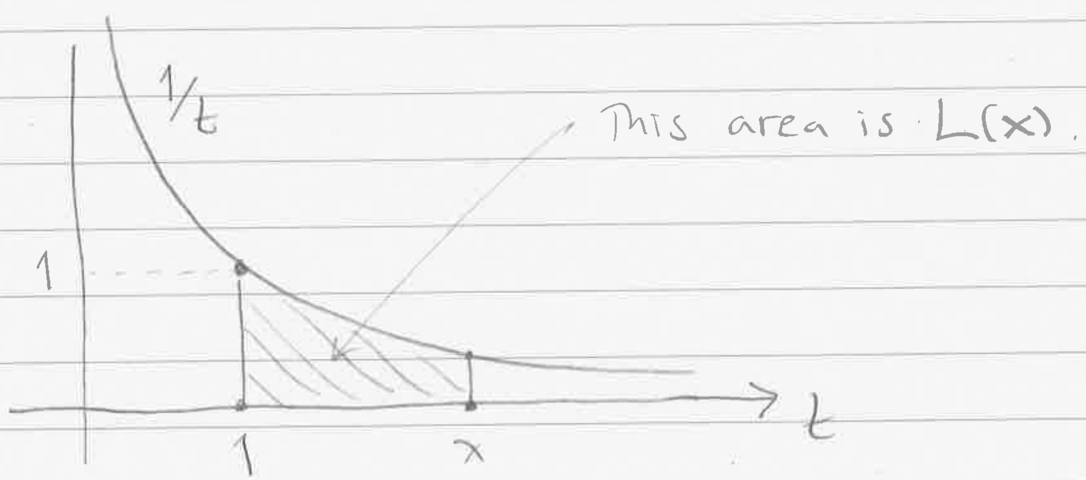
$$\text{“} \int x^{-1}\, dx = \frac{1}{0} x^{0} \text{”}$$

makes no sense. But certainly x^{-1} <u>does</u> <u>have</u> an antiderivative. Let's define

$$L(x) := \int_1^x \frac{1}{t}\, dt .$$

This is the area under the curve $1/t$ from $t=1$ to $t=x$.

Picture:



By Part 1 of the F.T.C. we have

$$L'(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

so $L(x)$ is an antiderivative of $1/x$. Our goal is to learn as much about this mysterious function $L(x)$ as we can.

What do we know so far?

- $L(1) = \int_1^1 1/t \, dt = 0$.
- $L(x) > 0$ when $1 < x$
  $L(x) < 0$ when $0 < x < 1$
- $L'(x) = 1/x$.

Maybe we can even sketch the graph of $L(x)$ because we know $L'(x) = 1/x$ and

$$L''(x) = (x^{-1})' = (-1)x^{-2} = -1/x^2.$$

Since $L'(x) = 1/x > 0$ when $x > 0$ we know that $L(x)$ is increasing. Since $L''(x) = -1/x^2 < 0$ when $x > 0$ we know that $L(x)$ is concave down.

So the graph looks something like this:

What else do we know?

Let $a > 0$ be constant and define the function $f(x) = L(ax)$. Then the Chain Rule says

$$f'(x) = L'(ax) \cdot (ax)'$$

$$= \frac{1}{ax} \cdot (\cancel{a}) = \frac{1}{x}.$$

In other words, $f(x) = L(ax)$ is an antiderivative of $1/x$. But we already know that $L(x)$ is an antiderivative of $1/x$, so we must have

$$L(ax) = L(x) + c$$

for some constant c. What is c? Plug in $x=1$ to get

$$L(a \cdot 1) = L(1) + c$$
$$L(a) = 0 + c.$$

We conclude that

$$\boxed{L(ax) = L(a) + L(x)}$$

Well that's rather strange. Have you ever seen a function like that? It turns multiplication into addition.

Such functions were invented in 1614 by John Napier and they are called "logarithms".

["L" is for "logarithm". Coincidence?]

Similarly we can compute the derivative of $L(x^r)$ where r is constant. We have

$$(L(x^r))' = L'(x^r) \cdot (x^r)'$$

$$= \frac{1}{x^r} \cdot r x^{r-1} = \frac{r}{x}$$

Dividing both sides by r gives

$$\left(\frac{1}{r} L(x^r)\right)' = \frac{1}{x}$$

so that $\frac{1}{r}L(x^r)$ is another antiderivative of $1/x$. We conclude that

$$\frac{1}{r} L(x^r) = L(x) + d$$

for some constant d. Plug in $x=1$ to get

$$\frac{1}{r} L(1^r) = L(1) + d.$$

$$\frac{1}{r} L(1) = L(1) + d$$

$$0 = 0 + d.$$

Finally we get

$$\frac{1}{r} L(x^r) = L(x) \;,$$

or $$\boxed{L(x^r) = r \cdot L(x)}$$

This is another property of logarithms. Putting together the two properties also gives

$$\begin{aligned} L\left(\frac{x}{y}\right) &= L(x \cdot y^{-1}) \\ &= L(x) + L(y^{-1}) \\ &= L(x) + (-1)\,L(y) \\ &= L(x) - L(y) \;. \end{aligned}$$

☆ In summary, our function $L(x)$ satisfies:

- $L(xy) = L(x) + L(y)$
- $L(x^r) = r \cdot L(x)$
- $L(x/y) = L(x) - L(y)$.

These are the characteristic properties of logarithms. But there are many different functions satisfying these properties. For each constant $b > 0$ there is a function $\log_b(x)$ (called the logarithm with base b) defined by

$$y = \log_b(x) \iff x = b^y.$$

The question remains: What is the base of our logarithm $L(x)$?

Assume that $L(x) = \log_b(x)$. We want to compute b. To do this first note that b is the unique number satisfying

$$L(b) = 1$$

Now we have a trick. We will compute $L'(1)$ in two ways.

On the one hand, $L'(1) = 1/1 = 1$.

On the other hand, the definition says

$$\downarrow$$

$$L'(1) = \lim_{h \to 0} \frac{L(1+h) - \cancelto{0}{L(1)}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot L(1+h)$$

$$= \lim_{h \to 0} L\left((1+h)^{1/h} \right)$$

$$= L\left(\lim_{h \to 0} (1+h)^{1/h} \right).$$

Since $1 = L'(1) = L\left(\lim_{h \to 0} (1+h)^{1/h} \right)$ we conclude that

$$b = \lim_{h \to 0} (1+h)^{1/h}.$$

OK, but what is the value of this limit? Luckily we've seen this before (on July 6). First let $n = 1/h$, so that

$$h \to 0 \quad \text{as} \quad n \to \infty.$$

Then we have

$$b = \lim_{h \to 0} (1+h)^{1/h}$$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \doteq 2.718\ldots$$

In summary, the mysterious function

$$L(x) = \int_1^x \frac{1}{t}\, dt$$

is actually the logarithm to the base e. This is called the "natural logarithm" and we use the notation

$$L(x) = \log_e(x) = \ln(x).$$

After all that work, we can say this:

$$\int x^p\, dx = \begin{cases} \dfrac{x^{p+1}}{p+1} & \text{if } p \neq -1 \\ \ln(x) & \text{if } p = -1. \end{cases}$$

7/27/15

Quiz 4 (25 minutes).
Return HW 4.

HW 4	Total	26
	Ave.	22.6
	Med	24
	St. Dev.	3.9

Now we're in the home stretch.

We have seen all the basic ideas of Calculus. This week we will learn some new functions and how to evaluate some harder integrals.

Next week we will see one or two cool applications of integrals and we will practice for the final exam.

Last day of class: Wed Aug 5
Final Exam: Fri Aug 7, 10:05 – 12:05.

First let's recall the discussion from Friday's class.

We defined the function

$$L(x) := \int_1^x \frac{1}{t}\, dt$$

By the F.T.C. we have $L'(x) = 1/x$.
Then we used the chain rule to prove the following properties:

- $L(x \cdot y) = L(x) + L(y)$
- $L(x^r) = r \cdot L(x)$
- $L(x/y) = L(x) - L(y)$.

These properties tell us that $L(x)$ is a Logorithmic function. This means that $L(x) = \log_b(x)$ for some b, where

$$y = \log_b(x) \iff x = b^y.$$

This b is called the "base" of the Logarithm.

Finally, we used a trick to show that

$$L(x) = \log_e(x)$$

where $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.71828\cdots$.

This is called the "natural logarithm" function and we use the special notation

$$L(x) = \log_e(x) = \ln(x).$$

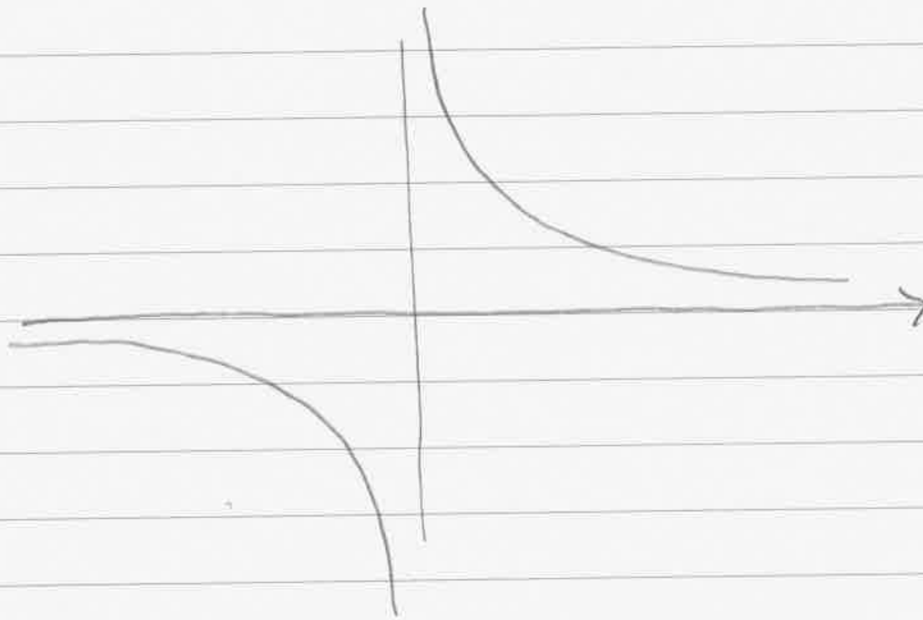
This allows us to state the full power law for antiderivatives:

$$\int x^p \, dx = \begin{cases} \frac{1}{p+1} \cdot x^{p+1} & \text{if } p \neq -1 \\ \ln(x) & \text{if } p = -1. \end{cases}$$

Actually, this is not quite true. The function $\ln(x)$ is not defined when $x < 0$, but the function $1/x$ <u>is</u>.

What can we do about this?

Here is the graph of $f(x) = 1/x$:



For $x > 0$ we know that the antiderivative is $\ln(x) + C$. What about when $x < 0$?

We will use a trick. Let $u = -x$ so that

$$f(x) = 1/x = -1/(-x) = -f(-x) = -f(u).$$

When $x < 0$, $u > 0$ so we should be able to compute the antiderivative of $-f(u)$.

we have

$$\int f(x)\,dx = \int -f(u)\,dx = -\int f(u)\,dx.$$

OK, but the $f(u)$ and dx don't match. We need to rewrite dx in terms of du.

$$u = -x$$

$$\frac{du}{dx} = -1$$

$$du = -dx.$$

So we get

$$\int f(x)\,dx = -\int f(u)\,dx$$

$$= -\int f(u)(-du)$$

$$= +\int f(u)\,du$$

$$= \ln(u) + C$$

$$= \ln(-x) + C.$$

And this <u>is</u> defined because $-x > 0$.

Check: Using the chain rule gives

$$\left(\ln(-x) + c\right)' = \frac{1}{-x}(-x)' + 0$$

$$= \frac{1}{-x}(-1) = \frac{1}{x} \quad \checkmark$$

In conclusion we have

$$\int \frac{1}{x}\,dx = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 . \end{cases}$$

or we can write this more efficiently:

$$\int \frac{1}{x}\,dx = \ln(|x|) + C ,$$

What about the general logarithmic function with base b?

$$y = \log_b(x) \iff x = b^y .$$

$\Downarrow$

Take the natural logarithm of both sides:

$$b^y = x$$

$$\ln(b^y) = \ln(x)$$

$$y \cdot \ln(b) = \ln(x)$$

$$\log_b(x) \cdot \ln(b) = \ln(x)$$

$$\boxed{\log_b(x) = \frac{\ln(x)}{\ln(b)}}$$

[Every logarithm can be expressed in terms of the natural logarithm.]

Now we can compute the derivative. Since $\ln(b)$ is just a constant, we have

$$\frac{d}{dx}\left(\log_b(x)\right) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(b)}\right)$$

$$= \frac{1}{\ln(b)} \cdot \frac{d}{dx}\left(\ln(x)\right) = \frac{1}{\ln(b)} \cdot \frac{1}{x}.$$

In summary:

$$\boxed{\frac{d}{dx}\left(\log_b(x)\right) = \frac{1}{x\cdot\ln(b)}}$$

Now let's look at "exponential functions".
We define the <u>natural</u> <u>exponential</u> as

$$\exp(x) := e^x.$$

Note that we have

$$e^x = y \iff x = \ln(y).$$

So that

- $\ln(e^x) = x$ for all x.
- $e^{\ln(x)} = x$ for $x > 0$.

This allows us to compute the derivative of $\exp(x)$.

Consider the equation

$$\ln(\exp(x)) = x.$$

Compute d/dx of both sides and use the chain rule to get

$$\frac{d}{dx}\left[\ln(\exp(x))\right] = \frac{d}{dx}x$$

$$\frac{1}{\exp(x)} \cdot \frac{d}{dx}\exp(x) = 1.$$

$$\boxed{\frac{d}{dx}\exp(x) = \exp(x)}$$

☆ The function $\exp(x) = e^x$ is equal to its own derivative. That's pretty interesting (and useful too). This fact is the foundation of the theory of <u>differential</u> <u>equations</u>.

Application: Suppose we are growing bacteria in a petri dish.

Let $p(t)$ be the number of bacteria at time t. If the rate of reproduction is $r > 0$ then the function $p(t)$ satisfies the differential equation

$$(*) \quad \boxed{p'(t) = r \cdot p(t)}$$

The general solution is

$$p(t) = c \cdot e^{rt}.$$

Proof: Using the chain rule gives

$$\begin{aligned} p'(t) &= (c \cdot e^{rt})' \\ &= c(e^{rt})' \\ &= c \cdot e^{rt} \cdot (rt)' \\ &= c \cdot e^{rt} \cdot r \\ &= r\,p(t). \end{aligned}$$

as desired. ///

The equation (*) governs the growth of a population under ideal conditions, the growth of money in a bank account, and the decay of radioactive material.

Finally, let's look at the general exponential function

$$\exp_b(x) = b^x$$

It has properties that are "inverse" to the properties of logarithms:

- $\exp_b(x+y) = \exp_b(x) \cdot \exp_b(y)$
- $\exp_b(x-y) = \exp_b(x) / \exp_b(y)$
- $\exp_b(r \cdot x) = \exp_b(x)^r$.

Exercise: Compute $\frac{d}{dx}\exp_b(x)$.

We have a trick for this.

Since $\exp(\ln(b)) = b$ we can write

$$\exp_b(x) = b^x = \left(e^{\ln(b)}\right)^x = e^{x \cdot \ln(b)}.$$

Now use the chain rule to get

$$\frac{d}{dx}\left(b^x\right) = \frac{d}{dx} e^{x\ln(b)}$$

$$= e^{x\ln(b)} \cdot \left(x\ln(b)\right)'$$

$$= \ln(b) \cdot e^{x\ln(b)}$$

$$= \ln(b) \cdot \left(e^{\ln(b)}\right)^x$$

$$= \ln(b) \cdot b^x.$$

Summary:

$$\boxed{\frac{d}{dx}\left(b^x\right) = \ln(b) \cdot b^x}$$

OK, that's enough formulas for today. We need practice.

Practice: Compute $\frac{d}{dx}$ of the following.

1. $x \cdot 5^x$

2. $\log_{10}(x^2)$

3. x^x

1. Here we use the product rule and the formula $(b^x)' = \ln(b) \cdot b^x$ to get

$$(x \cdot 5^x)' = (x)' \cdot 5^x + x \cdot (5^x)'$$

$$= 1 \cdot 5^x + x \cdot \ln(5) \cdot 5^x$$

$$= 5^x \left(1 + \ln(5) \cdot x \right).$$

2. Here we use the chain rule and the formula $(\log_b(x))' = 1/(\ln(b) \cdot x)$ to get

$$(\log_b(x^2))' = \frac{1}{\ln(b) \cdot x^2} \cdot (x^2)'$$

$$= \frac{1}{\ln(b) x^2} \cdot 2x = \frac{2}{\ln(b) \cdot x}.$$

3. This one is tricky. Many students would say

$$(x^x) = x \cdot x^{x-1}$$

but this is wrong because the exponent is not constant. Many students would say

$$(x^x)' = \ln(x) \cdot x^x$$

but this is wrong because the base is not constant.

Instead we will use our new favorite trick to write

$$x = e^{\ln(x)} .$$

so that

$$x^x = (e^{\ln(x)})^x = e^{x \cdot \ln(x)} .$$

Then we use the chain rule and product rule to get

$$
\begin{aligned}
(x^x)' &= \left(e^{x \cdot \ln(x)} \right)' \\
&= e^{x \cdot \ln(x)} \cdot \left(x \cdot \ln(x) \right)' \\
&= x^x \cdot \left(x \cdot \ln(x) \right)' \\
&= x^x \left[(x)' \cdot \ln(x) + x \cdot (\ln(x))' \right] \\
&= x^x \left[1 \cdot \ln(x) + \cancel{x} \cdot \frac{1}{\cancel{x}} \right] \\
&= x^x \left(\ln(x) + 1 \right).
\end{aligned}
$$

[Remark: The same trick can be used to compute the derivative of

$$f(x)^{g(x)}$$

for any functions $f(x)$ and $g(x)$.]

7/28/15

HW 5 due Friday.

Yesterday we discussed logarithmic and exponential functions. We developed some new formulas:

- $$\boxed{\frac{d}{dx}\left(\log_b(x)\right) = \frac{1}{x \cdot \ln(b)}}$$

Special Case: $b = e$ gives

$$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x \cdot \ln(e)} = \frac{1}{x}$$

because $\ln(e) = 1$.

- $$\boxed{\frac{d}{dx}\left(b^x\right) = \ln(b) \cdot b^x}$$

Special Case: $b = e$ gives

$$\frac{d}{dx}\left(e^x\right) = \ln(e) \cdot e^x = e^x$$

because $\ln(e) = 1$.

Reversing the second formula gives

$$(b^x)' = \ln(b) \cdot b^x$$

$$\int (b^x)' dx = \int \ln(b) \cdot b^x dx$$

$$b^x = \ln(b) \int b^x dx$$

$$\Longrightarrow \boxed{\int b^x dx = \frac{1}{\ln(b)} \cdot b^x + C}$$

But we still don't know how to reverse the first formula:

$$\int \log_b(x) dx = ?$$

Stay tuned . . .

We also learned a useful trick to compute the derivative of

$$f(x)^{g(x)}$$

where $f(x)$ & $g(x)$ are any functions.

TRICK: Rewrite $f(x)$ as

$$f(x) = e^{\ln(f(x))}.$$

so that

$$f(x)^{g(x)} = \left(e^{\ln(f(x))}\right)^{g(x)}$$

$$= e^{\ln(f(x)) \cdot g(x)}.$$

Then we can use the chain rule and product rule to compute

$$\left(f(x)^{g(x)}\right)' = \left(e^{\ln(f(x)) \cdot g(x)}\right)'$$

$$= e^{\ln(f(x)) \cdot g(x)} \cdot \left(\ln(f(x)) \cdot g(x)\right)'$$

$$= f(x)^{g(x)} \left[(\ln(f(x)))' \cdot g(x) + \ln(f(x)) \cdot g'(x) \right]$$

$$= f(x)^{g(x)} \cdot \left[\frac{1}{f(x)} \cdot f'(x) \cdot g(x) + \ln(f(x)) \cdot g'(x) \right]$$

Yes it's complicated but this is the correct answer.

Another way to describe the same computation is called "logarithmic differentiation". Let

$$y = f(x)^{g(x)}$$

Take the natural log of both sides:

$$\ln(y) = \ln\left(f(x)^{g(x)}\right) = g(x) \cdot \ln(f(x)).$$

Now compute d/dx of both sides:

$$\frac{1}{y} \cdot \frac{dy}{dx} = g(x) \cdot \left(\ln(f(x))\right)' + g'(x) \cdot \ln(f(x))$$

$$= g(x) \cdot \frac{1}{f(x)} \cdot f'(x) + g'(x) \cdot \ln(f(x)).$$

So that

$$\frac{dy}{dx} = y \left[\frac{g(x) f'(x)}{f(x)} + g'(x) \cdot \ln(f(x)) \right]$$

where $y = f(x)^{g(x)}$. This is the same answer as before.

Examples from Chap 5.4 Exercises:

Differentiate the following

23. $f(x) = x^5 + 5^x$

25. $f(t) = 10^{\sqrt{t}}$

33. $y = x^{\sin(x)}$.

23. $f'(x) = 5 \cdot x^4 + \ln(5) \cdot 5^x$

25. We will use the chain rule.

Let $u = \sqrt{t}$ so that $f(t) = 10^u$.
We want to compute df/dt. The chain rule says

$$\frac{df}{dt} = \frac{df}{du} \cdot \frac{du}{dt}$$

$$= \ln(10) \cdot 10^u \cdot \frac{1}{2} t^{-1/2}$$

$$= \ln(10) \cdot 10^{\sqrt{t}} \cdot \frac{1}{2} t^{-1/2}.$$

33. Let $y = x^{\sin(x)}$. We want to compute $\frac{dy}{dx}$.

We will use "logarithmic differentiation". First take the natural log of both sides.

$$\ln(y) = \ln\left(x^{\sin(x)}\right)$$

$$\ln(y) = \sin(x) \cdot \ln(x).$$

Now apply d/dx to both sides

$$\frac{1}{y} \cdot \frac{dy}{dx} = \cos(x) \cdot \ln(x) + \sin(x) \cdot \frac{1}{x}$$

Finally, solve for dy/dx.

$$\frac{dy}{dx} = y\left[\cos(x) \cdot \ln(x) + \sin(x) \cdot \frac{1}{x}\right]$$

$$= x^{\sin(x)}\left[\cos(x) \cdot \ln(x) + \sin(x) \cdot \frac{1}{x}\right]$$

☆ Logarithmic differentiation is often helpful to compute $f'(x)$ when $f(x)$ has a complicated formula.

///

We call logarithmic and exponential functions inverses of each other because

- $\log_b(\exp_b(x)) = x$
- $\exp_b(\log_b(x)) = x$.

In general, if $f(x)$ & $g(x)$ satisfy

- $f(g(x)) = x$
- $g(f(x)) = x$

then we say f & g are inverses.

Jargon: In this case we will write

$$f(x) = g^{-1}(x) \quad \& \quad g(x) = f^{-1}(x).$$

☆ WARNING:

$$f^{-1}(x) \neq \frac{1}{f(x)} \; !!!$$

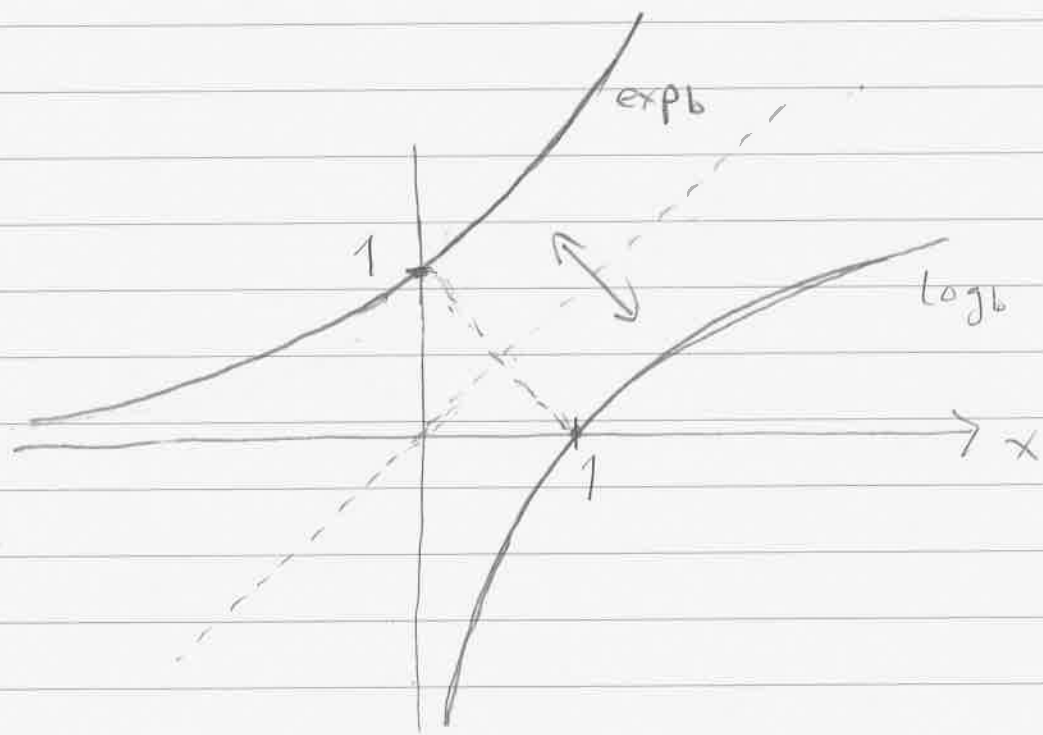
It's just the notation we use for the inverse function of $f(x)$.

Another way to say that $f(x) = g^{-1}(x)$ is

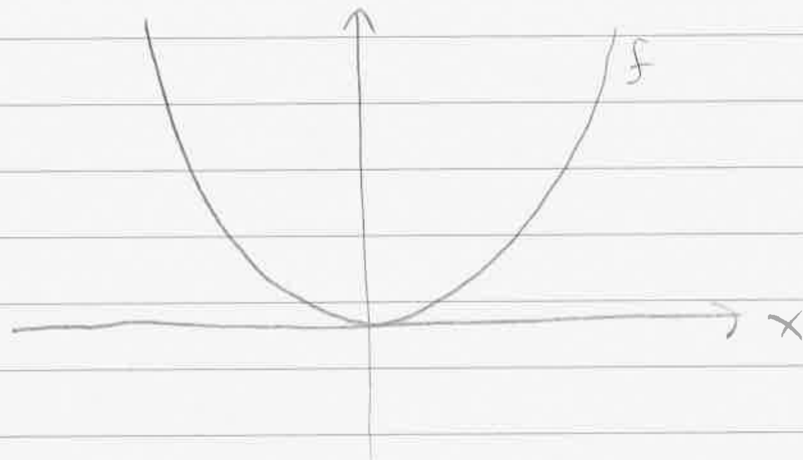
$$y = f(x) \iff x = g(y).$$

In terms of pictures, this tells us that to obtain the graph of $g(x)$ from the graph of $f(x)$ we should switch the x-axis and y-axis, i.e., we should <u>reflect</u> across the line $y = x$.

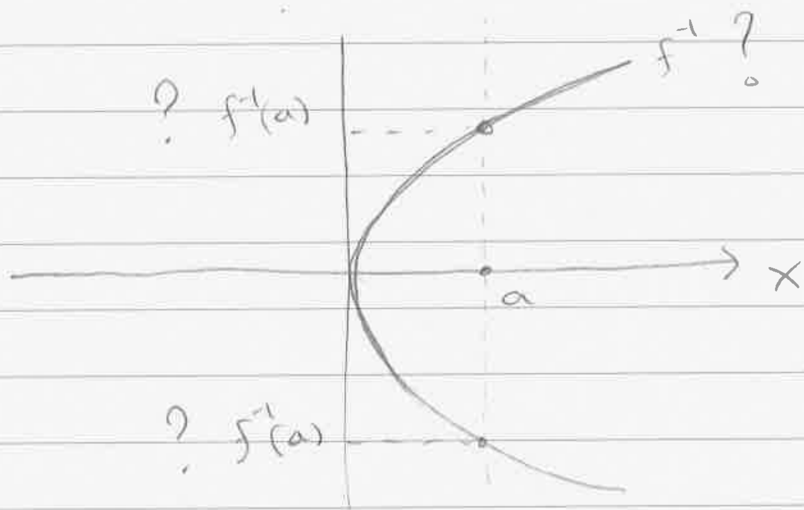
For example, here are the graphs of $\log_b(x)$ and $\exp_b(x) = b^x$:



But sometimes this doesn't work. For example, consider $f(x) = x^2$ with graph



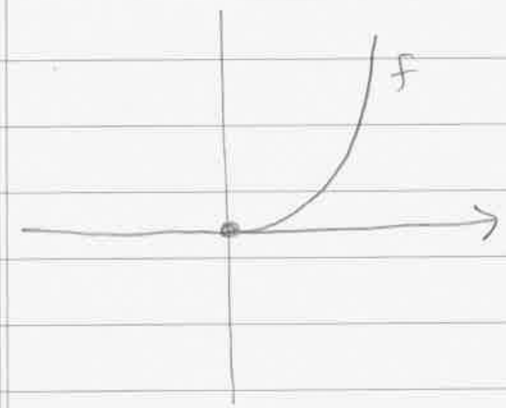
Then the graph of $f^{-1}(x)$ should look like

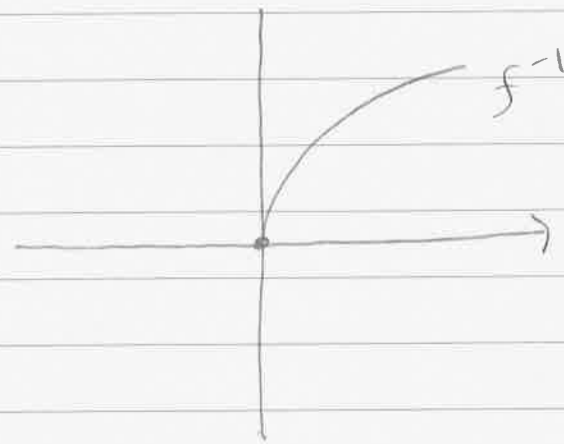


But we know that this isn't the graph of a function. [Remember the vertical line test?]

We say that the function $f(x) = x^2$ is <u>not</u> <u>invertible</u>.

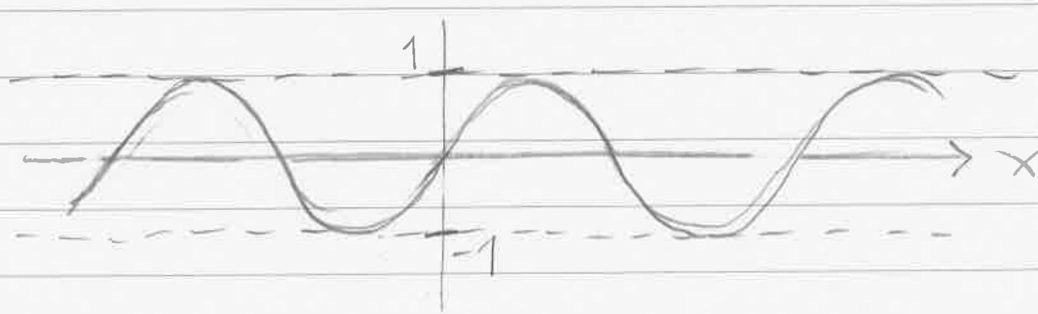
Sometimes we can fix this problem by "restricting the domain". For example, if we only consider $f(x) = x^2$ when $x > 0$ then it becomes invertible.



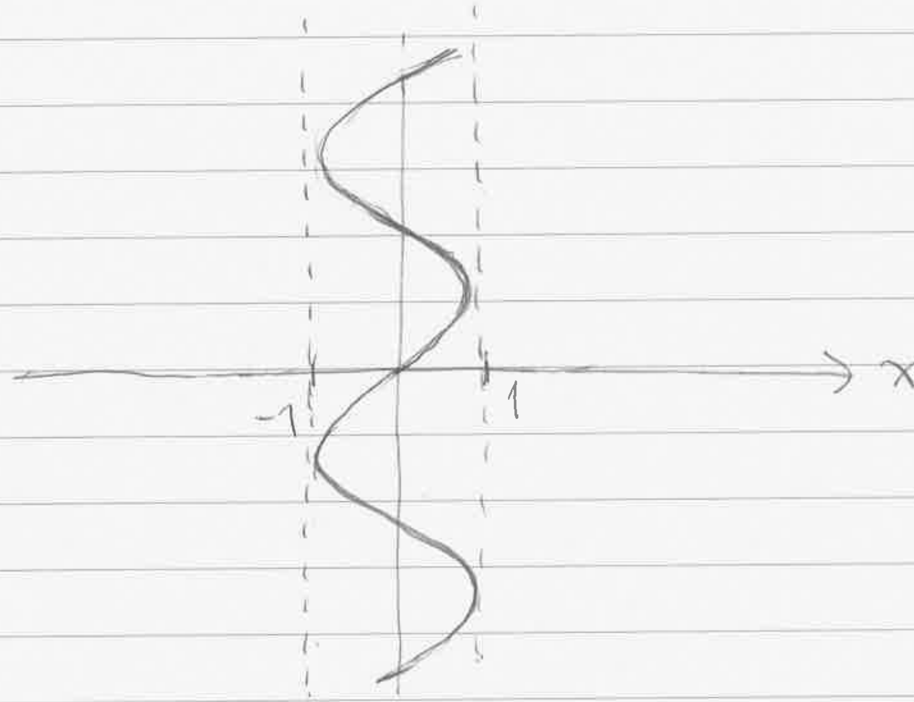


The inverse function is $f^{-1}(x) = +\sqrt{x}$, as you know.

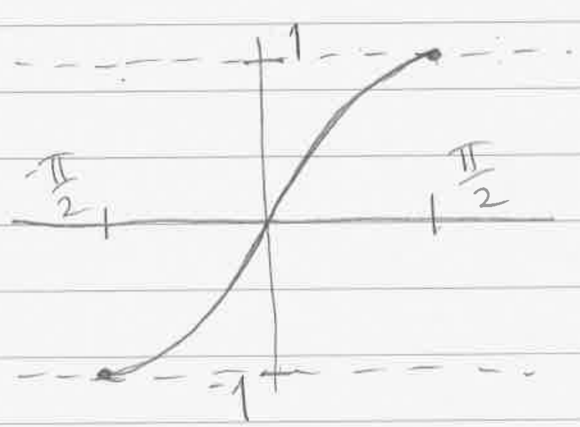
Let's apply the same idea to trig functions. The graph of $\sin(x)$ looks like



So the graph of $\sin^{-1}(x)$ should look like



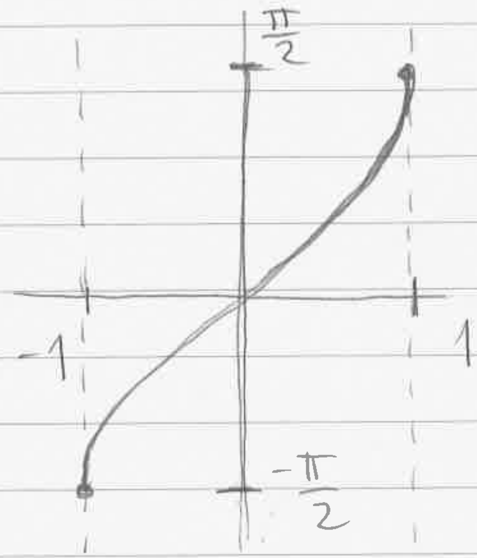
But this is certainly not the graph of a function. [It fails the vertical line test very badly.] To make $\sin^{-1}(x)$ into a function we must restrict the domain of $\sin(x)$, and the standard choice is to force $-\pi/2 \le x \le \pi/2$.



This is the graph of $\sin(x)$ restricted to

$$-\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

Now we can think of $\sin^{-1}(x)$ as a function with graph



☆ WARNING:

$$\sin^{-1}(x) \neq (\sin(x))^{-1}$$

$$\text{but } \sin^2(x) = (\sin(x))^2 \quad !!!$$

This is a really terrible notation but it is unfortunately fairly common. I prefer to write

$$\text{“}\arcsin(x) = \sin^{-1}(x)\text{”}$$

because it's less confusing.

Now within the proper ranges we can say

$$y = \arcsin(x) \iff x = \sin(y).$$

Our next job is to compute

$$\frac{dy}{dx} = \frac{d}{dx}\arcsin(x).$$

To do this we will apply d/dx to both sides of $x = \sin(y)$. We get

$$\frac{d}{dx}x = \frac{d}{dx}\sin(y)$$

$$1 = \cos(y)\cdot\frac{dy}{dx}$$

$$\Longrightarrow \quad \frac{dy}{dx} = \frac{1}{\cos(y)}$$

$$\Longrightarrow \quad \frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))}.$$

This is correct but maybe there's a way to simplify it?

Recall that $\sin(\arcsin(x)) = x$ by definition. Also recall that for all θ,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos\theta = \sqrt{1-\sin^2\theta}$$

Now put $\theta = \arcsin(x)$ to get

$$\cos(\arcsin(x)) = \sqrt{1-\sin^2(\arcsin(x))}$$

$$= \sqrt{1-x^2}.$$

We conclude that

$$\boxed{\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}}$$

and we choose the positive square root because the slope of $\arcsin(x)$ is positive (under our conventions).

Similarly, we can restrict the domains of $\cos(x)$ and $\tan(x)$ to define $\arccos(x)$ and $\arctan(x)$, and then we can compute:

- $\dfrac{d}{dx} \arccos(x) = \dfrac{-1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx} \arctan(x) = \dfrac{1}{1+x^2}$.

[I'll spare you the gory details.]

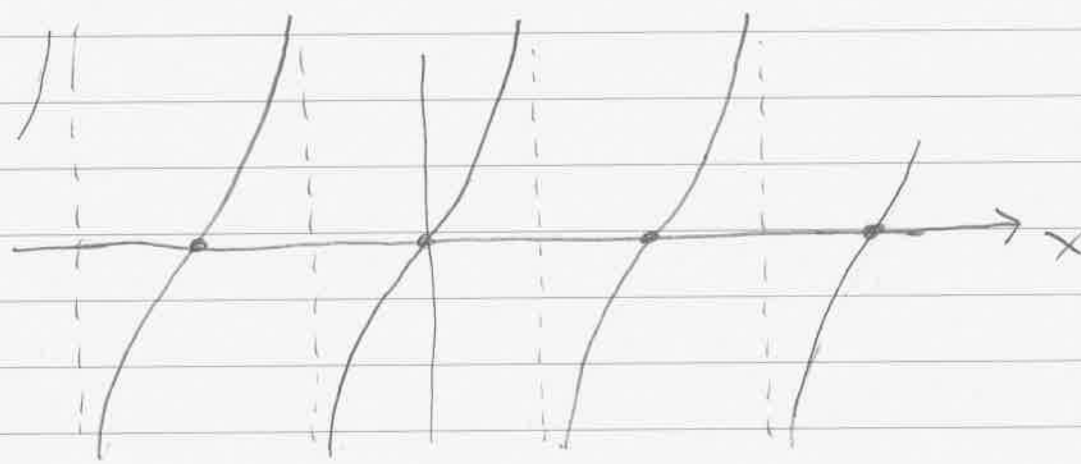
Actually we don't really care about inverse trig functions for their own sake. The <u>real</u> interest of these formulas is that they show us how to integrate some common functions:

- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$
- $\displaystyle\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos(x) + C$
- $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$.

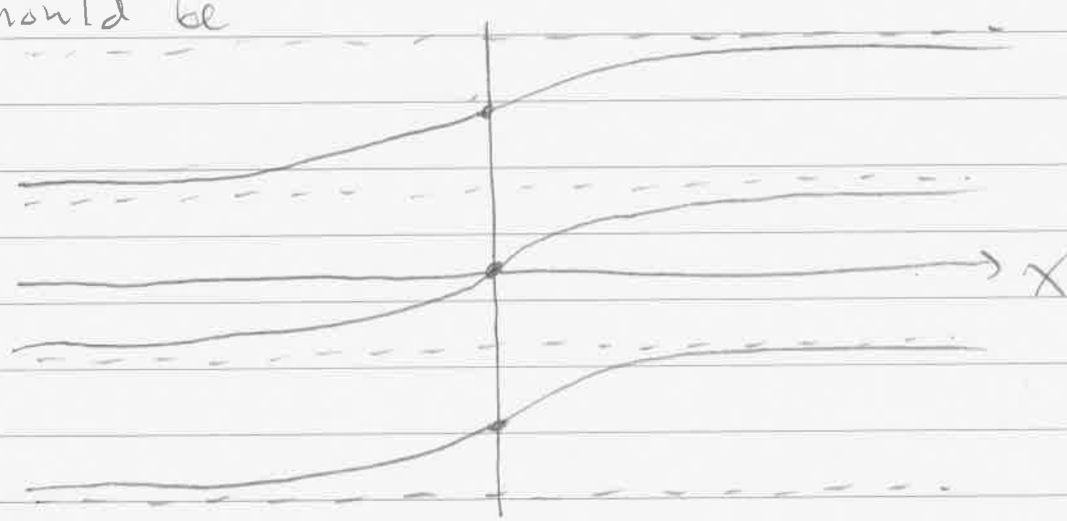
[Just like we had to invent $\ln(x)$ because we didn't know how to integrate $1/x$, we had to invent $\arctan(x)$ because we didn't know how to integrate $1/(1+x^2)$.]

Practice: Sketch the graphs of $\tan(x)$ & $\arctan(x)$.

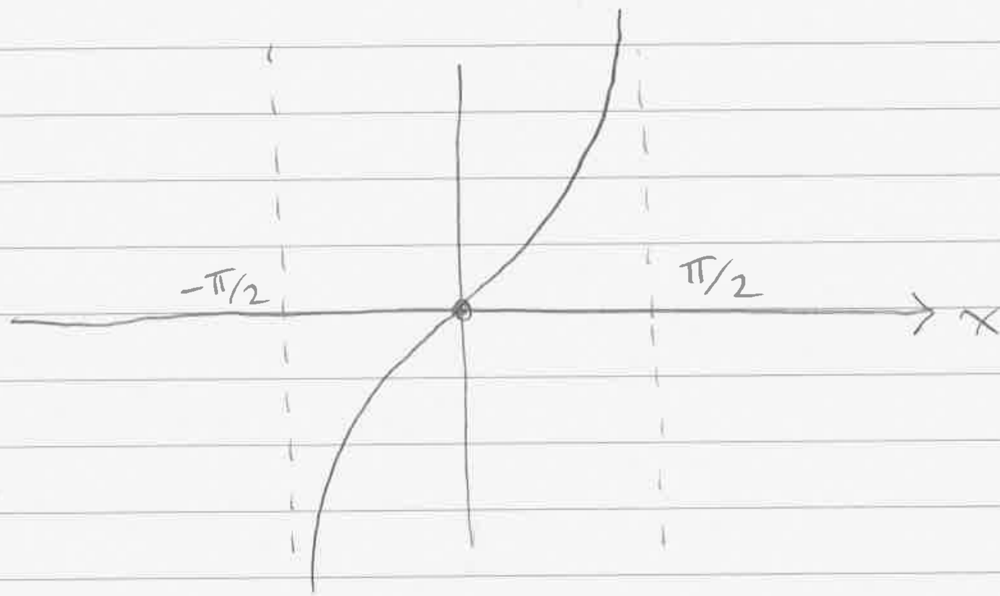
Here is the graph of $\tan(x)$.



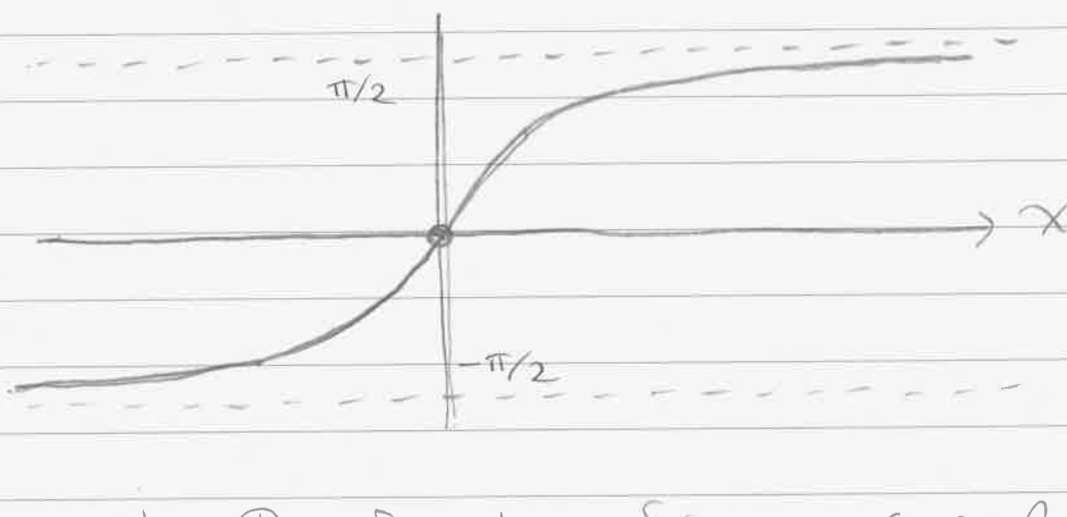
So the graph of $\arctan(x)$ $\left(= \tan^{-1}(x)\right)$ should be



But this is <u>not</u> the graph of a function. To get a function we must restrict the domain of $\tan(x)$. The natural choice is to restrict $-\pi/2 \le x \le \pi/2$:



Then the graph of $\arctan(x)$ is



[Remark: The function $f(x)$ on Quiz 2 Problem 2 was $f(x) = \arctan(2x)$.]

7/29/15

HW5 due Friday
Quiz 5 on Monday.
There will be <u>no</u> HW6 or Quiz 6.

Let's review the formulas we learned so far this week:

- $\int \frac{1}{x}\,dx = \ln|x| + C$

- $\frac{d}{dx}\log_b(x) = \frac{1}{x \cdot \ln(b)}$

- $\frac{d}{dx}(b^x) = \ln(b) \cdot b^x$

- $\int b^x\,dx = \frac{1}{\ln(b)} \cdot b^x + C$

- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$

- $\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

These are all rather specific. So far we haven't learned any "general purpose" techniques of integration.

Today we will.

Recall our "general purpose" rules for differentiation.

The Product Rule:

$$(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x).$$

The Chain Rule:

$$(f(g(x)))' = f'(g(x))\cdot g'(x).$$

We have seen that these are super useful. It would be nice to have similar rules for computing <u>anti</u>-derivatives.

Basically, we would like to run the chain rule and product rule in reverse, so that's what we'll do.

Let's start with the chain rule.

Example: Let $f(x) = \sin(-x^2)$.

Let's compute $f'(x)$ by making the substitution $u = -x^2$, so that $f = \sin(u)$. Then the chain rule says

$$f'(x) = \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

$$= \cos(u) \cdot (-2x)$$

$$= -2x \cdot \quad (-x^2).$$

OK, now compute the antiderivative

$$\int x \cdot \cos(-x^2)\, dx \, .$$

Hey, this isn't so bad because we just showed that

$$\int \left(-2x \cdot \cos(-x^2)\right) dx = \sin(-x^2) + C \, .$$

Divide both sides by -2 to get

$$\frac{1}{\cancel{-2}} \int \left(\cancel{-2}x \cos(-x^2)\right) dx = -\frac{\sin(-x^2)}{2} + \frac{C}{2}$$

$$\boxed{\int x \cos(-x^2)\, dx = -\frac{\sin(-x^2)}{2} + D}$$

Well OK. That was easy because we already knew the answer. What if we didn't already know the answer?

We would have to guess.

$$\int x \cos(-x^2)\, dx = ?$$

We think: Hmm... maybe this came from the chain rule. Let's try the substitution $u = -x^2$ and see what happens.

$$\int x \cos(-x^2)\, dx = \int x \cos(u)\, dx$$

$$= \int \cos(u)\, x\, dx.$$

Now we want to express dx in terms of du:

$$u = -x^2$$

$$du/dx = -2x$$

$$du = -2x\,dx$$

$$\frac{du}{-2} = x\,dx$$

Thus

$$\int x\cos(-x^2)\,dx = \int \cos(u)\,x\,dx$$

$$= \int \cos(u)\,\frac{du}{-2}$$

$$= -\frac{1}{2}\int \cos(u)\,du$$

$$= -\frac{1}{2}\sin(u) + C$$

$$= -\frac{1}{2}\sin(-x^2) + C.$$

///

That is called the method of substitution (though I would prefer to call it the inverse chain rule).

☆ Method of Substitution:

You are asked to compute $\int f(x)\,dx$. If $f(x)$ looks like it might be of the form $f(x) = g'(h(x)) \cdot h'(x)$ (i.e. if it looks like it came from the chain rule), then you should make the substitution $u = h(x)$ and see what happens.

[It's not really a rule, because it's not always easy to tell when to use it.

Slogan: Differentiation is a science; integration is an art.]

Examples from Chap 4.5 Exercises:

3. $\int x^2 \sqrt{x^3+1}\,dx = ?$

Use the substitution $u = x^3 + 1$.

so that $du = 3x^2\,dx$. Then we have

$$\int x^2\sqrt{x^3+1}\,dx = \int x^2\sqrt{u}\,dx$$

$$= \int \sqrt{u}\,x^2\,dx$$

$$= \int \sqrt{u}\cdot\frac{1}{3}\,du$$

$$= \frac{1}{3}\int \sqrt{u}\,du$$

$$= \frac{1}{3}\int u^{1/2}\,du$$

$$= \frac{1}{3}\cdot\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}\cdot\frac{2}{3}\left(x^3+1\right)^{3/2} + C$$

$$= \frac{2}{9}(x^3+1)^{3/2} + C.$$

It worked.

5. $\int \cos^3\theta \sin\theta\, d\theta = ?$

Make the substitution $u = \cos\theta$,

so that $du = -\sin\theta \, d\theta$. Then we have

$$\int \cos^3\theta \sin\theta \, d\theta = \int u^3 (-du)$$

$$= -\int u^3 \, du$$

$$= -\frac{1}{4}u^4 + C$$

$$= -\frac{1}{4}\cos^4\theta + C.$$

It worked, but how did we know to make the substitution $u = \cos\theta$? That's the hard part. In this case the textbook told us what to do. In real life you just have to guess.

18. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}} \, dx$.

No hint is given. So, what substiution should we use?

Maybe $u = \sqrt{x}$.

Will it work? We don't know. Let's try it and see.

Since $u = \sqrt{x}$ we have

$$du/dx = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\Longrightarrow dx = 2\sqrt{x}\, du.$$

Then we have

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx = \int \frac{\sin(u)}{\cancel{\sqrt{x}}} \cdot 2\cancel{\sqrt{x}}\, du$$

$$= 2\int \sin(u)\, du$$

$$= -2\cos(u) + C$$

$$= -2\cos\sqrt{x} + C.$$

It worked. Sometimes it won't. You just have to try.

What if we had been asked to compute

$$\int_1^4 \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx \ ?$$

There are two ways to do it. We could first compute

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = -2\cos\sqrt{x} + C$$

and then use the F.T.C. to get

$$\int_1^4 \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \left[-2\cos\sqrt{4} + C\right] - \left[-2\cos\sqrt{1} + C\right]$$

$$= -2\cos(2) + 2\cos(1).$$

OR, we could rewrite everything in terms of $u = \sqrt{x}$ to get

$$\int_{x=1}^{x=4} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int_{u=1}^{u=2} 2\sin(u)\,du$$

$$= -2\cos(u)\Big|_{u=1}^{u=2}$$

$$= -2\cos(2) - (-2\cos(1)).$$

Your choice. //

Here are two tricky examples where the substitution is NOT obvious.

Example 1: Compute $\int \frac{\ln(x)}{x}\,dx$.

$$u = ?$$

If we let $u = \ln(x)$ then $du/dx = 1/x$ and so $du = dx/x$. Then we have

$$\int \frac{\ln(x)}{x}\,dx = \int \frac{u}{x}\,dx$$

$$= \int u\,du$$

$$= \frac{1}{2}u^2 + C$$

$$= \frac{1}{2}(\ln(x))^2 + C.$$

Example 2: Compute $\int \tan(x)\,dx$

$$u = ?$$

Just try some stuff!

Eventually you might try $u = \cos x$, so

$$\frac{du}{dx} = -\sin x$$

$$dx = -\frac{du}{\sin x}$$

Then we have

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

$$= \int \frac{\cancel{\sin x}}{u} \cdot \left(\frac{-du}{\cancel{\sin x}} \right)$$

$$= -\int \frac{1}{u} \, du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C.$$

It would take a very long time to find this antiderivative if we didn't have the method of substitution!

P.S. The text book (pg. 267) says

$$\int \tan x \, dx = \ln|\sec x| + C.$$

What's going on?

Well, we have

$$-\ln(|\cos x|) = \ln(|\cos x|^{-1})$$

$$= \ln\left(\frac{1}{|\cos x|}\right)$$

$$= \ln\left(\left|\frac{1}{\cos x}\right|\right)$$

$$= \ln|\sec x|.$$

Thinking Problem:

$$\int \ln(x)\, dx = ?$$

Tomorrow we will find out how to solve this using the method of "integration by parts".

7/30/15

HW5 due tomorrow.
Quiz 5 on Monday.

We are in the process of learning how to integrate a wider variety of functions. Unfortunately, there is no good algorithm for this and some guesswork is often needed.

Yesterday we learned the technique of

☆ Integration by Substitution

If $f(x)$ has the form $g'(h(x)) \cdot h'(x)$ for some functions $g(x)$ and $h(x)$, then to compute $\int f(x)\,dx$ we should make the substitution $u = h(x)$.

[Unfortunately, it is not always easy to see that $f(x)$ has the form $g'(h(x)) \cdot h'(x)$.]

The method of substitution is just the reverse of the chain rule.

Recall that the chain rule says

$$\left(g(h(x))\right)' = g'(h(x)) \cdot h'(x) .$$

Take the antiderivative of both sides to get

$$\int \left(g(h(x))\right)' dx = \int g'(h(x)) \cdot h'(x)\, dx$$

$$\boxed{g(h(x)) + C = \int g'(h(x)) \cdot h'(x)\, dx}.$$

This equation just describes the method of substitution.

Today we will develop another "general purpose" technique of integration by running the product rule in reverse.

Recall that the product rule says

$$\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Now take the antiderivative of both sides to get

$$\int \left(f(x)\cdot g(x)\right)' dx = \int \left(f'(x)g(x) + f(x)g'(x)\right) dx$$

$$f(x)\cdot g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

We can rearrange this to get

$$\boxed{\int f(x)g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\,dx}$$

This is the formula for "integration by parts", although I would prefer to call it the <u>inverse</u> <u>product</u> <u>rule</u>.

Example: Compute $\int x\cdot e^x\,dx$ using integration by parts.

We want to express $x\cdot e^x = f(x)\cdot g'(x)$, but which one is $f(x)$ and which one is $g(x)$?

Let's try it both ways:

First way: Let $f(x) = x$ and $g'(x) = e^x$
so that $f'(x) = 1$ and $g(x) = e^x$.

Then we have

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x) g(x)\, dx$$

$$\int x \cdot e^x\, dx = x \cdot e^x - \int 1 \cdot e^x\, dx$$

$$= x \cdot e^x - e^x + C.$$

It worked.

Second way: Let $f(x) = e^x$ and $g'(x) = x$
so that $f'(x) = e^x$ and $g(x) = x^2/2$.

Then we have

$$\int x \cdot e^x\, dx = \frac{1}{2} x^2 \cdot e^x - \int \frac{1}{2} x^2 \cdot e^x\, dx$$

$$= \frac{1}{2} x^2 e^x - \frac{1}{2} \int x^2 \cdot e^x\, dx.$$

Well, this is true but maybe not very helpful. We actually made it look more complicated.

The lesson: Choose $f(x)$ and $g'(x)$ so that

- You know how to compute $f'(x)$ and $g(x)$.
- $f'(x) \cdot g(x)$ is simpler than $f(x) \cdot g'(x)$.

You will often see the formula for integration by parts written in Leibniz notation as follows:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example: Compute $\int t\sin(t)\,dt$.

We have two options here.

- $u = t$ and $dv = \sin t\,dt$
- $u = \sin t$ and $dv = t\,dt$.

Which should we choose?

We want $v\,du$ to look simpler than $u\,dv$.

- $u = t$ and $dv = \sin t\, dt$ mean

$du = dt$ and $v = -\cos t \implies v\,du = -\cos t\, dt$.

- $u = \sin t$ and $dv = t\,dt$ mean

$du = \cos t\, dt$ and $v = \frac{1}{2}t^2 \implies v\,du = \frac{1}{2}t^2 \cos t\, dt$.

So we choose the first one:

$$\int u\,dv = uv - \int v\,du$$

$$\int t \sin t\, dt = -t\cos t - \int (-\cos t)\,dt$$

$$= -t\cos t + \sin t + C$$

Check:

$$(-t\cos t + \sin t + C)'$$

$$= (-t)'\cos t + (-t)(\cos t)' + (\sin t)' + 0$$

$$= -\cancel{\cos t} + t\sin t + \cancel{\cos t} = t\sin t$$

Practice: Compute $\int x^2 \cdot e^x \, dx$ by parts.

Let $f(x) = x^2$, $g'(x) = e^x$ so that
$f'(x) = 2x$, $g(x) = e^x$.

Then we have

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$$
$$= x^2 e^x - 2 \int x \cdot e^x \, dx.$$

And we already know $\int x \cdot e^x \, dx = xe^x - e^x$.
So we have

$$\int x^2 e^x \, dx = x^2 e^x - 2(xe^x - e^x) + C$$
$$= x^2 e^x - 2x e^x + 2e^x + C$$
$$= (x^2 - 2x + 2) e^x + C.$$

The Lesson: Sometimes we have to do integration by parts multiple times.

Recall: Yesterday we solved

$$\int \cos^3\theta \sin\theta \, d\theta$$

using the substitution $u = \cos\theta$, so that $du = -\sin\theta\, d\theta$. We got

$$\int \cos^3\theta \sin\theta \, d\theta = \int u^3 \sin\theta \, d\theta$$
$$= \int u^4 (-du)$$
$$= -\int u^4 \, du$$
$$= -\frac{1}{4} u^4 + C$$
$$= -\frac{1}{4} \cos^4\theta + C$$

Well, we could also solve it using integration by parts.

$$\cos^3\theta \sin\theta = f(\theta)\, g'(\theta)$$

Here we only have one reasonable choice because we don't know the anti-derivative of $\cos^3\theta$. So let

$$f(\theta) = \cos^3\theta \quad \& \quad g'(\theta) = \sin\theta$$
$$f'(\theta) = 3\cos^2\theta \cdot \sin\theta, \quad g(\theta) = -\cos\theta$$

Then we have

$$\int f(\theta)\, g'(\theta)\, d\theta = f(\theta) g(\theta) - \int f'(\theta)\, g(\theta)\, d\theta$$

$$\int \cos^3\theta \sin\theta \, d\theta = -\cos^4\theta - \int -3\cos^3\theta \sin\theta \, d\theta$$

$$= -\cos^4\theta + 3 \int \cos^3\theta \sin\theta \, d\theta .$$

Wait, it doesn't look any simpler. We just got the same integral back again. But that's actually good.

Let $A = \int \cos^3\theta \sin\theta \, d\theta$.

Then we have

$$A = -\cos^4\theta + 3 \cdot A$$

$$4A = -\cos^4\theta$$

$$A = -\frac{1}{4}\cos^4\theta .$$

See? We got the right answer. Which method do you like better?

Now we can solve a mystery that's been bugging us for a few days:

$$\int \ln(x)\, dx = ?$$

The trick is to think $\ln(x) = f(x)\, g'(x)$ for some $f(x)$ and $g'(x)$....

Aha! Let $f(x) = \ln(x)$ and $g'(x) = 1$ so that $f'(x) = 1/x$ and $g(x) = x$.

Then we have

$$\int f(x)\, g'(x) = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$$

$$\int \ln(x) \cdot 1\, dx = x \cdot \ln(x) - \int \frac{1}{\cancel{x}} \cdot \cancel{x}\, dx$$

$$\int \ln(x)\, dx = x \cdot \ln(x) - \int 1\, dx$$

$$\boxed{\int \ln(x)\, dx = x \cdot \ln(x) - x + C}$$

Interpreting $\ln(x)$ as $\ln(x) \cdot 1$ was a nice trick.

Practice: Use the same trick to compute

$$\int \arctan(x)\, dx.$$

We let $\arctan(x) = f(x)\, g'(x)$ where $f(x) = \arctan(x)$ and $g'(x) = 1$. Then

$$f'(x) = \frac{1}{1+x^2} \quad \text{and} \quad g(x) = x.$$

We have

$$\int \arctan(x)\, dx = x \cdot \arctan(x) - \int \frac{x}{1+x^2}\, dx.$$

Ok, now we have to solve

$$\int \frac{x}{1+x^2}\, dx.$$

We use the substitution $u = 1+x^2$, so that $du = 2x\, dx$ to get

$$\int \frac{x}{1+x^2}\, dx = \int \frac{x}{u} \cdot dx$$

$$= \int \frac{1}{u} \cdot x\, dx$$

$$= \int \frac{1}{u} \cdot \frac{du}{2}$$

$$= \frac{1}{2} \int \frac{1}{u}\, du$$

$$= \frac{1}{2} \ln|u| + C$$

$$= \frac{1}{2} \ln|1+x^2| + C.$$

Finally, we conclude that

$$\boxed{\int \arctan(x)\, dx = x \cdot \arctan(x) - \frac{1}{2}\ln|1+x^2| + C}$$

[I put it in a box, but actually I would prefer that you <u>don't</u> waste mental space memorizing this formula.]

Of course, integration by parts can also be used to evaluate <u>definite</u> integrals. The formula is

$$\int_a^b f(x)\, g'(x)\, dx = f(x) \cdot g(x) \Big|_a^b - \int_a^b f'(x) g(x)\, dx$$

where $f(x)g(x)\Big|_a^b = f(b)g(b) - f(a)g(a)$

But I don't think we need to practice that, do we?

We have now seen enough examples to observe a general phenomenon:

"The antiderivative of $f(x)$ is usually more complicated than $f(x)$."

This is in contrast to derivatives where $f'(x)$ is usually simpler than $f(x)$, or at least it's the same kind of function.

This makes the subject of symbolic integration frankly kind of ridiculous. It's mostly a grab bag of tricks with limited utility. Certainly it's useful to know the most common kinds of antiderivatives, but after that we should leave integration to the computers.

In the real world, computers are used to find approximate values for integrals, and that's good enough.

For example, it is known that the following antiderivatives exist, but they cannot be expressed in terms of any functions we know:

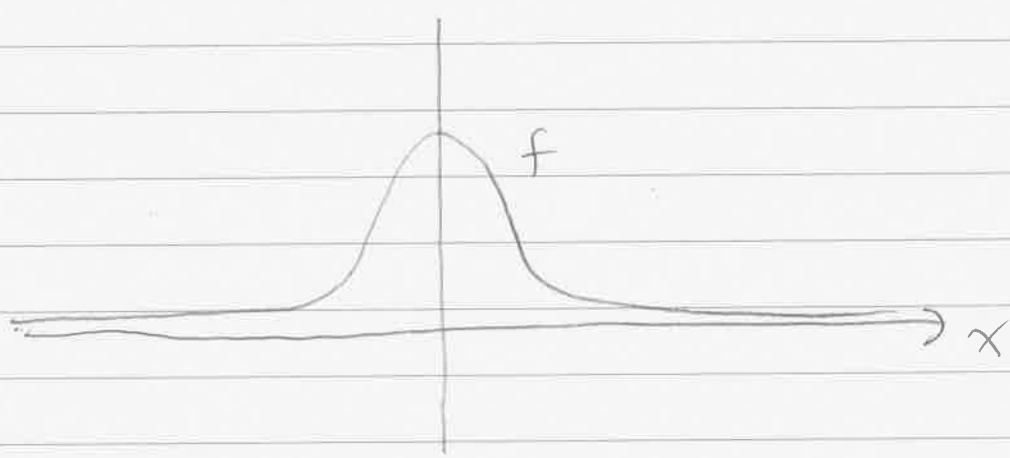
$$\int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx$$

$$\int \sqrt{x^3+1}\,dx \qquad \int \frac{1}{\ln(x)}\,dx \qquad \int \frac{\sin x}{x}\,dx .$$

Maybe the most important of these kind of functions is

$$N(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} .$$

It is easy to sketch the graph of $f(x)$ (see Chap 5.3 Exercise 60):



This is called the "bell-shaped curve" or the "Gaussian distribution" or just the "normal distribution".

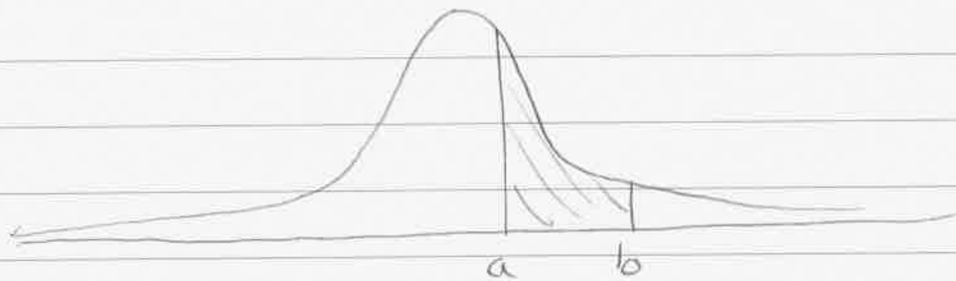
One can show (with a clever trick from Calc. III) that

$$\int_{-\infty}^{\infty} N(x) = 1 .$$

i.e., the total area under the whole curve is 1. This makes the function very useful in probability and statistics

In fact, there is a fancy theorem that tells us that if you repeat any random process often enough, its distribution will tend to a normal distribution.

Therefore is is very important to be able to compute specific areas under the normal curve:



$$\int_a^b N(x)\,dx = ?$$

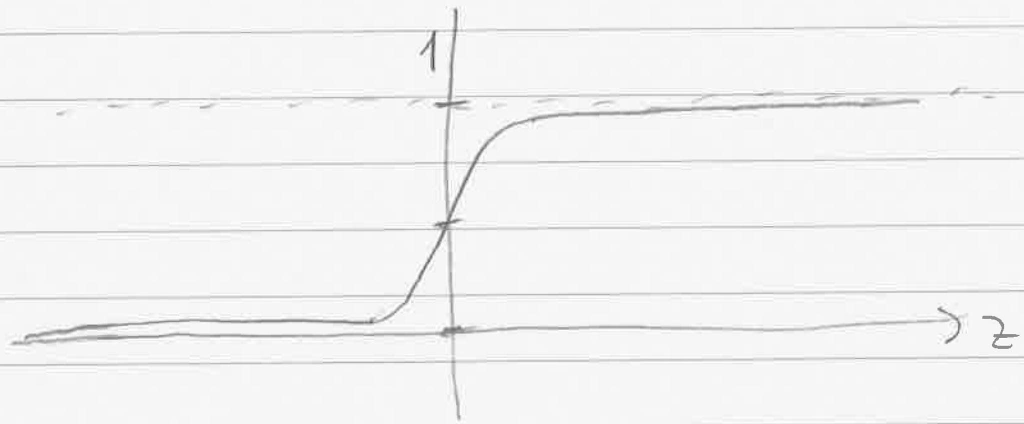
The standard way to do this is to define the function

$$\Phi(z) = \int_{-\infty}^{z} N(x)\,dx .$$

Since $\Phi'(x) = N(x)$ we have

$$\int_a^b N(x)\,dx = \Phi(b) - \Phi(a)$$

This $\Phi(z)$ is a perfectly nice function. We can easily sketch its graph:



But unfortunately $\Phi(z)$ cannot be expressed in terms of elementary functions. The best we ~~can~~ do is to give it a name. We define the "error function"

$$\mathrm{erf}(z) := 2\left[\Phi(z) - \frac{1}{2}\right]$$

Your calculator may even have a button to compute this. Every statistics book in the world has a table of values for $\Phi(z)$ in the back.