7/6/15

HW1 due NOW.
Quiz 1 NOW (25 minutes).

Reminder: My Office Hours are
Tues / Thurs at 1–2p in Mem. 216.
Also the Math Lab (Merrick 304)
is open most days until 6pm.

- Discuss HW1 and Quiz 1 Solutions.
- Discuss two hard limits.

1. Last week I asked you to consider

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n .$$

What did you find?

This limit has the form "1^∞" which is another indeterminate form. The value depends on <u>how quickly</u> the exponent approaches ∞ and <u>how quickly</u> the base approaches 1.

What can we do?

Let $f(n) = \left(1+\frac{1}{n}\right)^n$ and plug in values:

$$
\begin{aligned}
f(1) &= 2 \\
f(10) &= 2.593742460 \\
f(100) &= 2.704813829 \\
f(1000) &= 2.716923932 \\
f(10000) &= 2.718145927 \\
f(100000) &= 2.718268273
\end{aligned}
$$

Is this approaching a number we recognize?

Yes it is approaching a number, but maybe not one we recognize. This is a new number and there is no simpler way to describe it so we just make a definition.

Definition: We define the number

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n .$$

$$e \approx 2.7182818284 6 \ldots$$

This number first appeared in 1618 and it was called "e" by Leonhard Euler (1727). We keep this name in his honor.

OK, so it doesn't have a simpler expression, but is there a simple way to <u>think</u> about e?

Yes. You put \$1 in a bank account with 100% yearly interest. How much will you have in 1 year?

\$2?

But what if the interest is calculated twice during the year?

$$\$1\left(1+\tfrac{1}{2}\right)\left(1+\tfrac{1}{2}\right) = \$1.50\left(1+\tfrac{1}{2}\right)$$
$$= \$2.25.$$

Hey, that's better! Let's calculate the interest three times!

$$\$1\left(1+\tfrac{1}{3}\right)\left(1+\tfrac{1}{3}\right)\left(1+\tfrac{1}{3}\right) = \$1.33\left(1+\tfrac{1}{3}\right)\left(1+\tfrac{1}{3}\right)$$
$$= \$1.77\left(1+\tfrac{1}{3}\right)$$
$$= \$2.37$$

Now we get greedy. Ask the bank to compute your interest every second.

There are 31536000 seconds in the non-leap year.

The amount of money in your bank account at the end of the year is

$$\$1\left(1+\frac{1}{31536000}\right)^{31536000} = \$2.71828\cdots$$
$$= \$2.72$$

rounded to the nearest cent.

No matter how often the bank computes interest, the best you will ever do is

$$\lim_{n\to\infty} \$1\left(1+\frac{1}{n}\right)^n = \$1 \cdot \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= \$1 \cdot e$$

$$= \$e.$$

///

On HW2 you will show that \$1 in a bank account with yearly rate of return r becomes

$$\$ e^{r}$$

after 1 year.

- Last Thursday I wrote

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = ?$$

on the board. Did you think about it?

This limit is impossible to solve by algebraic means. It requires some geometric reasoning.

There is an argument on pg 42 of Stewart and pg. 63 of Spivak showing that

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$

Instead of following their argument, I'll refer to HW1 Problem 2. There we gave a convincing argument that

$$\lim_{n \to \infty} n \cdot \tan\left(\frac{\pi}{n}\right) = \pi.$$

But recall that $\tan\theta = \sin\theta / \cos\theta$, hence

$$\lim_{n \to \infty} n \cdot \tan\left(\frac{\pi}{n}\right) = \lim_{n \to \infty} \frac{n \cdot \sin\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n}\right)}$$

$$= \frac{\lim_{n \to \infty} n \cdot \sin\left(\frac{\pi}{n}\right)}{\underbrace{\cancel{\lim_{n \to \infty} \cos\left(\frac{\pi}{n}\right)}}_{1}}$$

$$= \lim_{n \to \infty} n \cdot \sin\left(\frac{\pi}{n}\right).$$

Hence $\lim_{n \to \infty} n \cdot \sin\left(\frac{\pi}{n}\right) = \pi$ also.

How does this relate to $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$?

TRICK: We can make the substitution

$$\boxed{\theta = \frac{\pi}{n}}.$$

Note that as $n \to \infty$ we have $\theta \to 0^+$ (from the right). Hence,

$$\pi = \lim_{n \to 0} n \cdot \sin\left(\frac{\pi}{n}\right)$$

$$= \lim_{\theta \to 0^+} \frac{\pi}{\theta} \cdot \sin\theta$$

$$= \pi \cdot \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta}.$$

Dividing both sides by π then gives

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1.$$

[Exercise: What about $\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta}$?]

The method of substitution that we used here is very useful. We will see it again.

Recitation:

1. $\lim\limits_{x \to 0} \dfrac{\sin(3x)}{x} = ?$

2. $\lim\limits_{x \to 0} \dfrac{\sin(4x)}{\sin(6x)} = ?$

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1. This limit has the form "0/0" so we need a trick. We will use the method of substitution to turn it into a limit we recognize.

Let $\theta = 3x$. Then we have

$$\theta \to 0 \quad \text{as} \quad x \to 0$$

so that

$$\lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{\theta\to 0} \frac{\sin\theta}{(\theta/3)}$$

$$= \lim_{\theta\to 0} 3\cdot\frac{\sin\theta}{\theta}$$

$$= 3\cdot\left(\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\right)$$

$$= 3\cdot 1 = 3.$$

2. This limit has the form "0/0" so we need a trick. This time it's harder to imagine how to make the limit look like $\lim_{\theta\to 0} \sin\theta/\theta$.

Here's how to do it. Mutiply top and bottom by x to get

$$\lim_{x\to 0} \frac{\sin(4x)}{\sin(6x)} = \lim_{x\to 0} \frac{x\cdot\sin(4x)}{x\cdot\sin(6x)}$$

$$= \lim_{x\to 0} \frac{x}{\sin(6x)}\cdot\frac{\sin(4x)}{x}$$

$$= \lim_{x \to 0} \frac{x}{\sin(6x)} \cdot \lim_{x \to 0} \frac{\sin(4x)}{x}$$

Now we will compute both of these limits separately. Using the substitution $\theta = 4x$ gives

$$\lim_{x \to 0} \frac{\sin(4x)}{x} = \lim_{\theta \to 0} \frac{\sin\theta}{(\theta/4)}$$

$$= \lim_{\theta \to 0} 4 \cdot \frac{\sin\theta}{\theta}$$

$$= 4 \cdot \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 4 \cdot 1 = 4.$$

Next we'll let $t = 6x$ to get

$$\lim_{x \to 0} \frac{x}{\sin(6x)} = \lim_{t \to 0} \frac{t/6}{\sin t}$$

$$= \frac{1}{6} \cdot \lim_{t \to 0} \frac{t}{\sin t}$$

$$= \frac{1}{6} \cdot \lim_{t \to 0} \frac{1}{t/\sin t}$$

$$= \frac{1}{6} \cdot \frac{1}{1} = \frac{1}{6}.$$

Finally we conclude that

$$\lim_{x \to 0} \frac{\sin(4x)}{\sin(6x)}$$

$$= \lim_{x \to 0} \frac{x}{\sin(6x)} \cdot \lim_{x \to 0} \frac{\sin(4x)}{x}$$

$$= \frac{1}{6} \cdot 4 = \frac{4}{6} = \frac{2}{3}.$$

///

[WARNING: In general we have

$$\sin(nx) \neq n \cdot \sin(x).$$

However, if $x \approx 0$ then we do have

$$\sin(nx) \approx n \cdot \sin(x).$$

Be careful!]

7/7/15.

Now we are done our whirlwind tour of the theory of limits and we must dive into Chapters 2 & 3:

"Derivatives & Applications".

The two pillars of Calculus are the theories of derivatives & integrals. They are connected by the Fundamental Theorem of Calculus.

F.T.C.

Derivatives	$\longleftrightarrow$	Integrals
"Differential Calculus"		"Integral Calculus"

We saw the idea of integrals in the introduction. They are used to compute the areas & volumes of strange shapes.

But what is a "derivative"?

The idea of a derivative is deeply related to physics. This is what Isaac Newton was thinking about in the 1660s.

Problem: An apple falls from a height of 100 feet. When will it hit the ground?

Before we can answer this we must discuss position, velocity & acceleration.

Let $s(t)$ be the height of the apple above the ground at time t. If we know the function $s(t)$, how can we compute the velocity of the apple? By definition,

$$\boxed{\text{velocity} = \frac{\text{distance traveled}}{\text{time elapsed}}}$$

For example, consider two times

$$t_1 < t_2$$

The distance traveled by the apple from time t_1 to time t_2 is

$$s(t_2) - s(t_1) = (\text{final position}) - (\text{initial position})$$

The amount of time elapsed is $t_2 - t_1$. Therefore the <u>average</u> <u>velocity</u> of the apple between times t_1 & t_2 is

$$(*) \quad \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

OK, fine. But how can we compute the <u>instantaneous</u> <u>velocity</u> of the apple at a <u>single</u> moment of time t?

What would that even mean?!

If we try to use formula (*) we get

$$\text{velocity at time } t = \frac{s(t) - s(t)}{t - t} = \text{``}\frac{0}{0}\text{''}$$

NONSENSE.

The only way to make sense of the concept of "the instantaneous velocity at time t" is as a limit.

First we compute the average velocity from time t to time $t+h$ (where h is variable) and then we compute the limit as $h \to 0$.

$$\text{average velocity from } t \text{ to } t+h. = \frac{s(t+h) - s(t)}{(\cancel{t}+h) - \cancel{t}}$$

$$= \frac{s(t+h) - s(t)}{h}$$

$$\text{instantaneous velocity at } t = \lim_{h \to 0} \frac{s(t+h) - s(t)}{t}.$$

[This is why we care about limits.]

There are a lot of different notations surrounding this idea that we should learn to recognize.

Notation 1 : Given a function $s(t)$ we define another function $s'(t)$ by

$$\boxed{s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(h)}{h}}$$

If $s(t)$ is the position of a particle (maybe an apple) at time t, then $s'(t)$ is the instantaneous velocity of the particle at time t.

Newton used the notation

$$\dot{s}(t) = s'(t)$$

and called it a "Fluxion".

Notation 2 : Given a function $s(t)$ we will let $\Delta s = s(t_2) - s(t_1)$ denote the change in s over a certain interval of time $\Delta t = t_2 - t_1$. The "average velocity" during this time is

$$\frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

[Remark: Δ is the Greek letter D.
"D" stands for "Difference".]

To compute the "instantaneous velocity" we consider the limit of $\Delta s / \Delta t$ as $\Delta t \to 0$.

Leibniz used the notation

$$\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

and he called this the "derivative of s with respect to t". We can think of ds as a tiny (infinitesimal) change in s and dt as a tiny change in t. These ds and dt are not really numbers, so BE CAREFUL.

Both Newton's and Leibniz' notations are in use today but the word "Fluxion" is no longer used. Everyone calls it the "derivative".

Example: Suppose the height (in feet) of an apple at time t (in seconds) is

$$s(t) = 100 - 16t^2$$

(a) Compute the average velocity between $t_1 = 1.8$ and $t_2 = 2.1$ seconds.

(b) Compute the instantaneous velocity at time $t = 2$ seconds.

(c) When will the apple hit the ground?

(d) How fast is it going when it does?

(a) The average velocity is

$$\frac{s(2.1) - s(1.8)}{2.1 - 1.8} = \frac{(\cancel{100} - 16(2.1)^2) - (\cancel{100} - 16(1.8)^2)}{0.2}$$

$$= -16 \left[\frac{(2.1)^2 - (1.8)^2}{0.3} \right]$$

$$= -62.4 \text{ feet/second.}$$

[The negative sign means the apple is going down.]

(b) Intantaneous velocity at $t=2$ is

$$\lim_{h \to 0} \frac{s(2+h)-s(2)}{(\cancel{2}+h)-\cancel{2}}$$

$$= \lim_{h \to 0} \frac{(\cancel{100}-16(2+h)^2)-(\cancel{100}-16\cdot 2^2)}{h}$$

$$= \lim_{h \to 0} \frac{-16}{h}\left[(2+h)^2-2^2\right]$$

$$= \lim_{h \to 0} \frac{-16}{h}\left[\cancel{4}+4h+h^2-\cancel{4}\right]$$

$$= \lim_{h \to 0} \frac{-16}{h}(4h+h^2)$$

$$= \lim_{h \to 0} \frac{-16\,\cancel{h}(4+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} -16(4+h) = -16\cdot 4 = -64$$

feet/sec.

(c) The apple hits the ground when

$$s(t) = 0$$
$$100 - 16t^2 = 0$$
$$100 = 16t^2$$
$$100/16 = t^2.$$

So $t = +\sqrt{100/16}$

$$= +\sqrt{100} / +\sqrt{16}$$
$$= 10/4.$$
$$= 2.5 \text{ seconds}.$$

[The other answer $t = -2.5$ is not relevant in this problem.].

(d) So we never have to do it again, let's compute the instantaneous velocity at a general time t.

$$s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{(\cancel{t}+h) - \cancel{t}}$$

$$= \lim_{h \to 0} \frac{(\cancel{100} - 16(t+h)^2) - (\cancel{100} - 16t^2)}{h}$$

$$= -16 \cdot \lim_{h \to 0} \frac{(t+h)^2 - t^2}{h}$$

$$= -16 \cdot \lim_{h \to 0} \frac{(\cancel{t^2} + 2th + h^2) - \cancel{t^2}}{h}$$

$$= -16 \cdot \lim_{h \to 0} \frac{2th + h^2}{h}$$

$$= -16 \cdot \lim_{h \to 0} \frac{\cancel{h}(2t+h)}{\cancel{h}}$$

$$= -16 \cdot \lim_{h \to 0} (2t+h)$$

$$= -16 \cdot (2t+0) = -32t .$$

In summary, we have

$$s(t) = 100 - 16 \cdot t^2$$
$$= \text{height at time } t .$$

$$\frac{ds}{dt} = s'(t) = -32 \cdot t$$
$$= \text{instantaneous velocity at time } t .$$

When the apple hits the ground (at $t=2.5$) its velocity is

$$s'(2.5) = -32\cdot(2.5) = -80$$

feet/second.

Notice that the apple is speeding up. How can we measure the change of velocity?

Let's compute the derivative of the derivative! Let's call it $s''(t)$.

$$s''(t) = \lim_{h\to 0} \frac{s'(t+h)-s'(t)}{h}$$

$$= \lim_{h\to 0} \frac{-32(\cancel{t}+h)+\cancel{32\cdot t}}{h}$$

$$= \lim_{h\to 0} \frac{-32\cdot \cancel{h}}{\cancel{h}}$$

$$= \lim_{h\to 0} -32 = -32 \text{ (ft/sec)/sec.}$$

The second derivative of position (i.e. the first derivative of velocity) is called the instantaneous acceleration.

It is measured in units (feet/second)/second, or feet/second^2.

Galileo Galilei observed (1638) that all bodies fall with a constant acceleration of ≈ -32 feet/sec^2.

So our calculation

$$s''(t) = -32$$

is physically plausible.

[Remark: The Leibniz notation for the "2nd derivative" is

$$\frac{d(ds/dt)}{dt} = \frac{d^2 s}{(dt)^2}.$$

Don't take this literally!]

7/8/15

HW 2 due Friday in class.
Quiz 2 Monday in class

Right now we are discussing Chapters 2 & 3 : "Derivatives and Applications".

Let $s(t)$ be the position of a particle at time t. Last time we defined the instantaneous velocity of the particle at time t to be

$$s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{(t+h) - t}$$

$$= \lim_{t_2 \to t_1} \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

$$= \text{“} \frac{ds}{dt} \text{”}$$

We called $s'(t)$ the derivative of the function $s(t)$.

The "2nd derivative" of $s(t)$,

$$s''(t) = \lim_{h \to 0} \frac{s'(t+h) - s'(t)}{(t+h) - t}$$

is called the (instantaneous) acceleration of the particle at time t.

If $s(t)$ is the height of a falling object (near the surface of the earth), Galileo observed that the acceleration is always constant (i.e. independent of time):

$$s''(t) = -32 \text{ feet}/\text{sec}^2$$

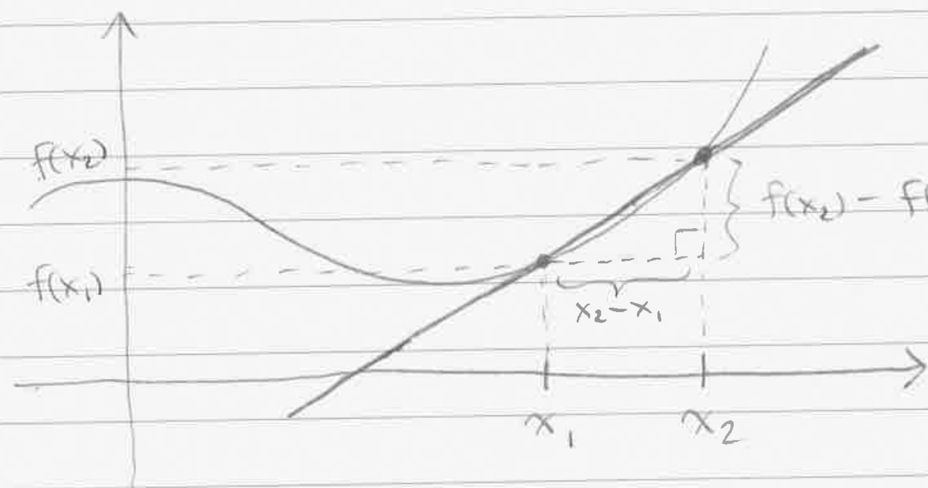
$$= 9.81 \text{ meters}/\text{sec}^2$$

[Remark: This constant is often called g, the acceleration due to gravity.]

Based on this observation, we could solve any problem about falling bodies.

But the derivative also has a geometric interpretation.

Let $f(x)$ be any function of a real variable x and consider its graph



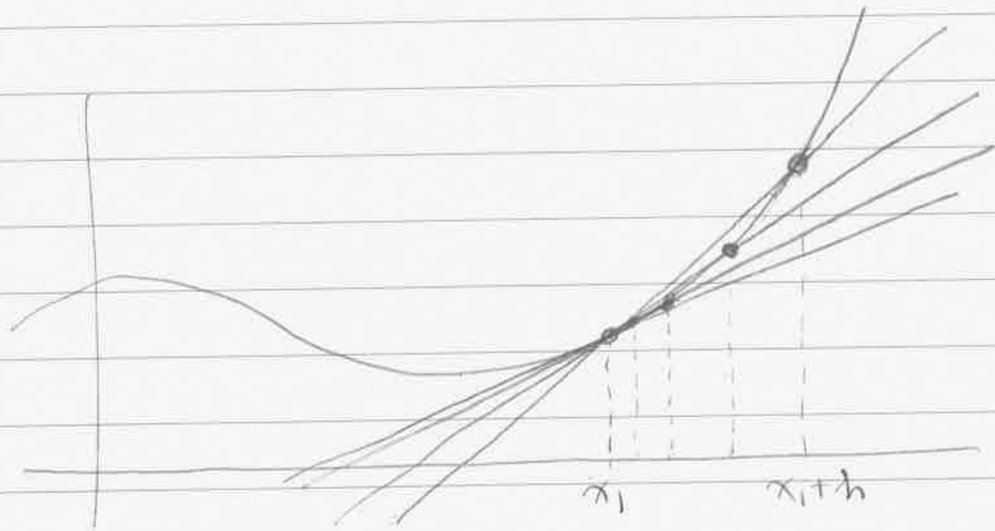
If we pick two x values, say $x_1 < x_2$, then the "difference ratio"

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

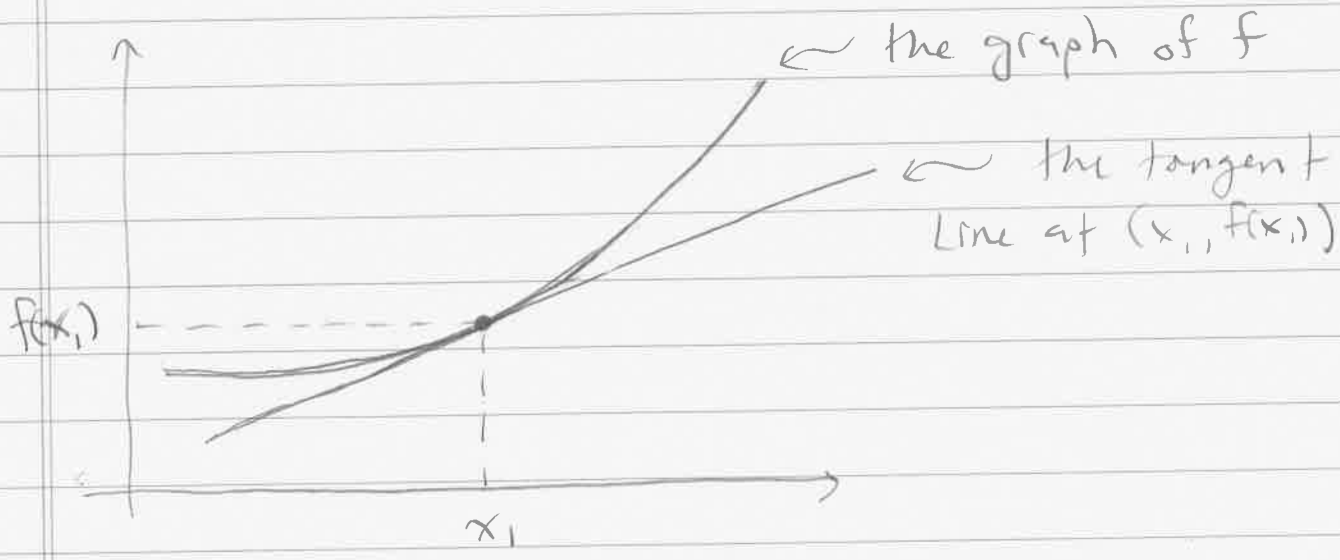
represents the slope of the line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Sometimes we call this a "secant line".

What happens if we take the values of x_1 & x_2 closer and closer together?

Say $x_2 = x_1 + h$ and we take $h \to 0$.



As $h \to 0$, the "secant line" becomes a line that is "tangent" to the curve at the point $(x_1, f(x_1))$.



What is the slope of the tangent line?

To compute slope we need to know two points on the line and we only know one!

What can we do?

The only way to make sense of the slope of the tangent line is as a limit.

By definition, the slope of the tangent at the point x_1 is

$$\lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{slope of a secant line}$$

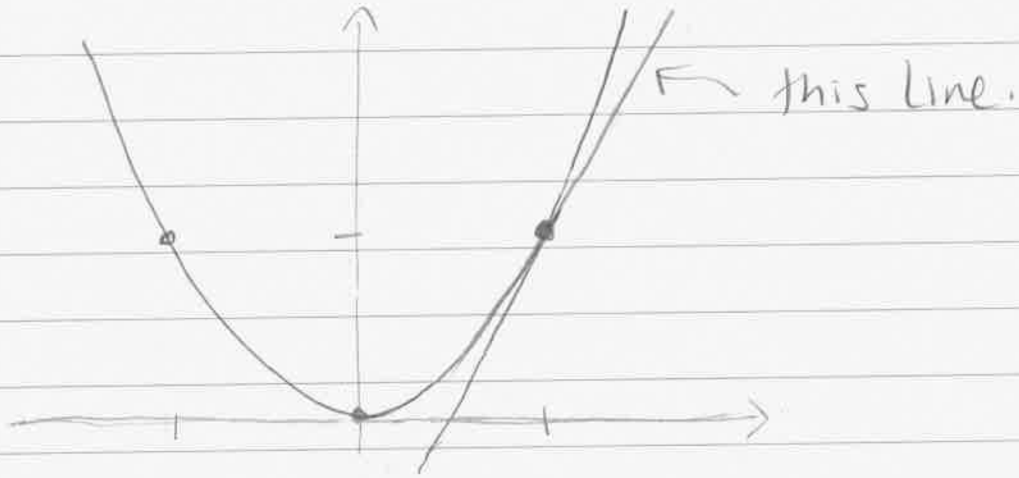
$$= \lim_{h \to 0} \frac{f(x_1+h) - f(x_1)}{(x_1+h) - x_1}$$

$$= f'(x_1).$$

$=$ the derivative of f at x_1

This is the geometric meaning of the derivative.

Example: Find the equation of the tangent line to the graph of $f(x) = x^2$ at the point $(1, 1)$.



$$f'(x) = \lim_{h \to 0} \left[\text{slope of the secant connecting } (x+h, f(x+h)) \text{ and } (x, f(x)) \right]$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (2x + h) = 2x .$$

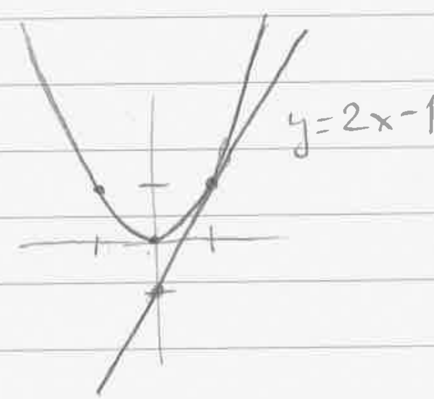
The slope of the tangent at (x, x^2) is $2x$.

So the slope of the tangent at $(1,1)$ is $2 \cdot 1 = 2$.

What is the equation of the line?

We have a line of slope 2 passing through the point $(1,1)$. The "point-slope" formula gives equation

$$2 = \frac{y-1}{x-1}$$



$$2(x-1) = y-1$$

$$2x - 2 + 1 = y$$

$$\boxed{2x - 1 = y.}$$

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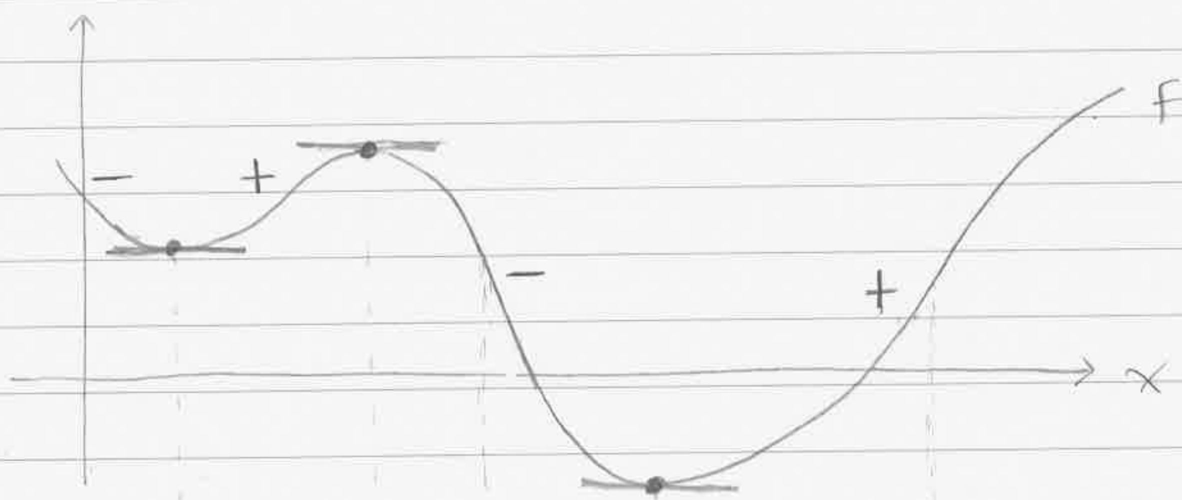
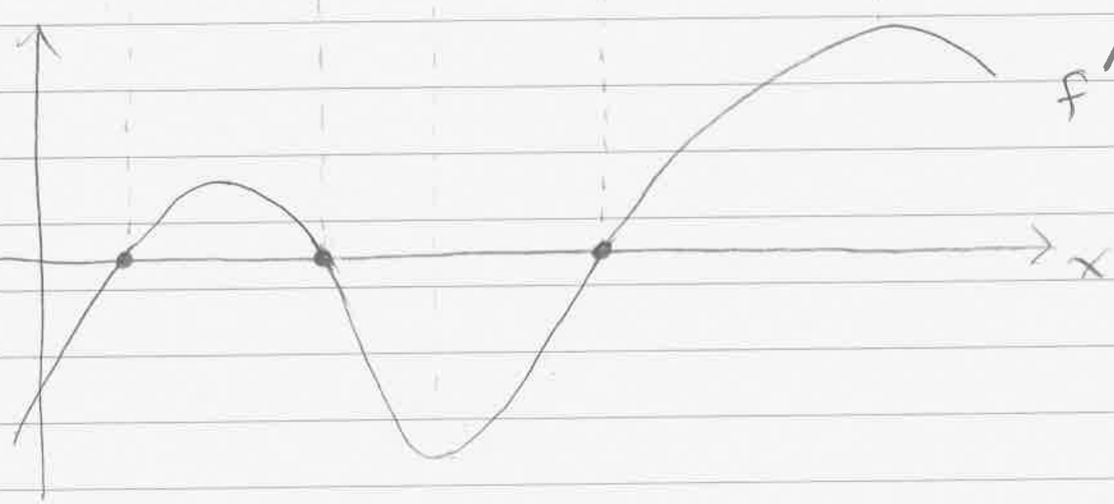
Sometimes we can make a rough sketch of the derivative function without doing any calculations.

Example: The following curve is the graph of $f(x)$. Use this to sketch the graph of the derivative $f'(x)$.





Here, f' shows the slope of the tangent to f.

Observations:

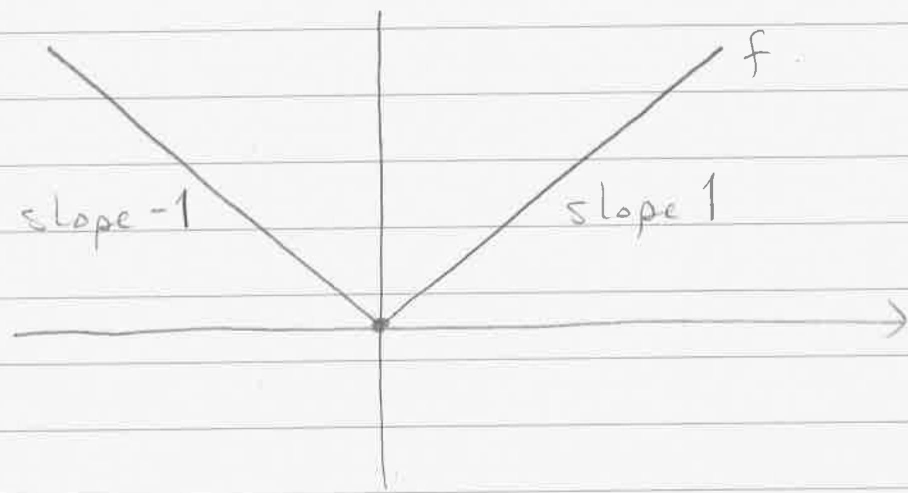
- When f has a "Local maximum" or "local mininum" at x, the tangent line is horizontal and hence $f'(x) = 0$.
- If f is increasing at x then the tangent has positive slope, so $f'(x) > 0$.
- If f is decreasing at x then the tangent has negative slope, so $f'(x) < 0$

Another Example: Draw the graph of the absolute value function

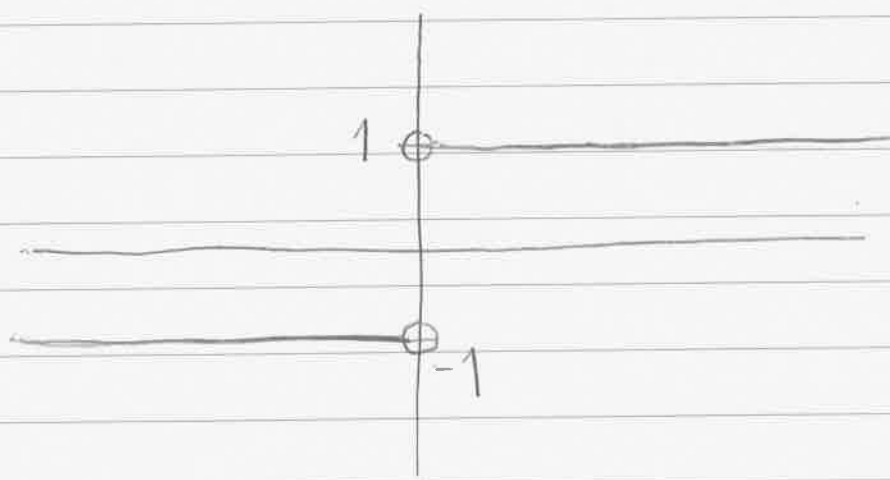
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

and use this to compute its derivative.

Let $f(x) = |x|$. Its graph is



The graph of $f'(x)$ is



So we have

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Jargon: We say that the function $f(x) = |x|$ is not differentiable at $x = 0$. This is because the graph has no natural tangent line at this point.

We can also sketch the graph of f''.

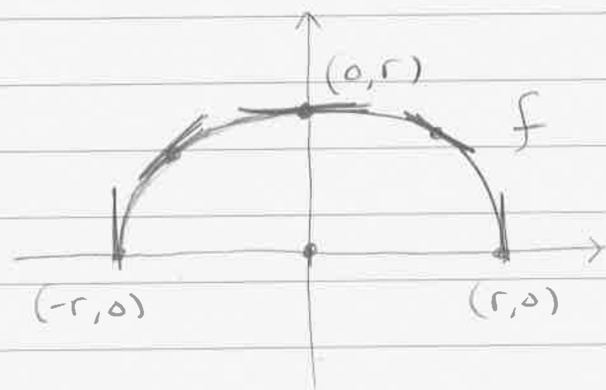
Example

- f has local max or min when $f' = 0$
- f has an "inflection point" when $f'' = 0$
- f' has local max or min when $f'' = 0$.

[In this case it looks like $f'''(x) = c$ for some positive constant $c > 0$, and hence $f''''(x) = 0$ is the zero function.]

Recitation:

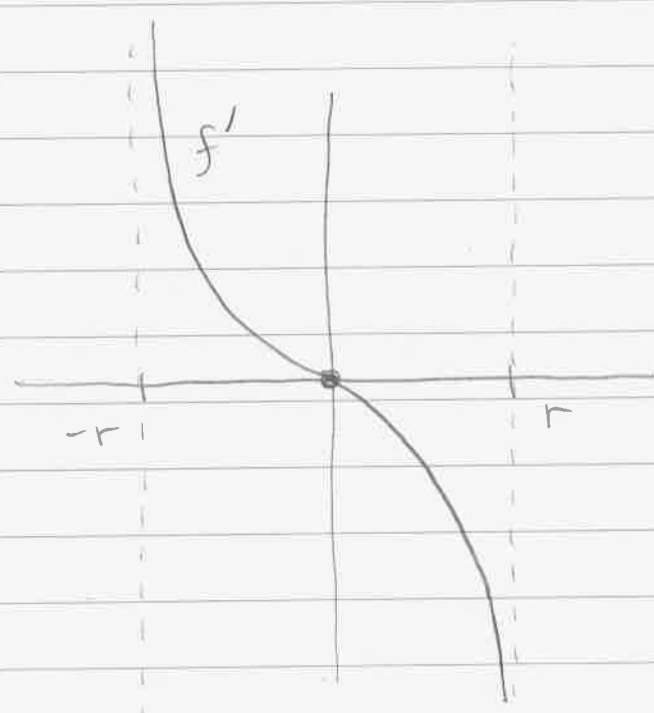
Consider the function $f(x)$ whose graph is an upper semicircle.



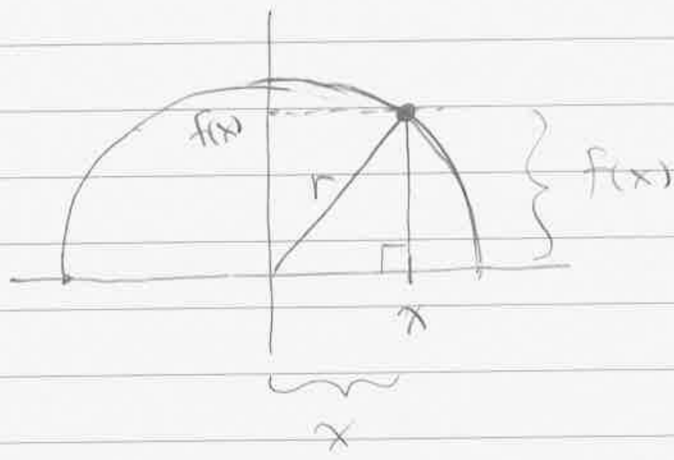
(a) Sketch the graph of $f'(x)$.

(b) Find formulas for both $f(x)$ and $f'(x)$.

(a)



(b) We can use the Pythagorean Theorem:



$$\boxed{x^2 + f(x)^2 = r^2}$$

Hence $x^2 + f(x)^2 = r^2$

$$f(x)^2 = r^2 - x^2$$

$$f(x) = +\sqrt{r^2 - x^2}.$$

[We choose the positive square root for the upper half of the circle.]

Now we can use the definition to compute $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{r^2 - (x+h)^2} - \sqrt{r^2 - x^2}}{h}$$

["rationalize" the numerator]

$$= \lim_{h \to 0} \frac{\sqrt{r^2-(x+h)^2} - \sqrt{r^2-x^2}}{h} \left(\frac{\sqrt{r^2-(x+h)^2} + \sqrt{r^2-x^2}}{\sqrt{r^2-(x+h)^2} + \sqrt{r^2-x^2}} \right)$$

$$= \lim_{h \to 0} \frac{(r^2-(x+h)^2) - (r^2-x^2)}{h\left(\sqrt{r^2-(x+h)^2} + \sqrt{r^2-x^2}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{r^2} - \cancel{x^2} - 2xh - h^2 - \cancel{r^2} + \cancel{x^2}}{h\left(\sqrt{r^2-(x+h)^2} + \sqrt{r^2-x^2}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h}\left(\sqrt{r^2-(x+h)^2} + \sqrt{r^2-x^2}\right)}$$

$$= \frac{-2x}{\sqrt{r^2-x^2} + \sqrt{r^2-x^2}} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{r^2-x^2}}$$

$$= -x \Big/ \sqrt{r^2-x^2} .$$ ///

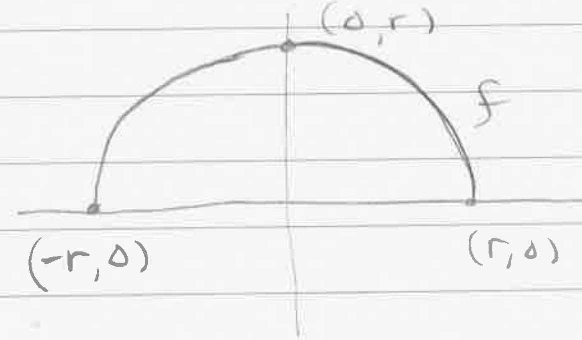
[Later we will have a shortcut for this sort of calculation.]

7/9/15

HW 2 due Friday in class
Quiz 2 Monday in class.

Last time we considered the function

$f(x) = \sqrt{r^2 - x^2}$



and we computed its derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

⋮

LOTS OF WORK

⋮

$$= -x / \sqrt{r^2 - x^2}.$$

Are you tired of computing derivatives the long way? Today we will develop some tricks and shortcuts

↓

that will turn the computation of derivatives into a routine. This is the "calculation" part of "calculus".

We will begin with the simplest functions and build up to more complicated examples.

The simplest function is a constant.

☆ Claim: If $f(x) = c$ is <u>constant</u>, then

$$f'(x) = 0.$$

Proof: We have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = 0. \quad ///$$

[Idea: A constant doesn't change, so its rate of change is zero.]

☆ Claim: If $f(x)$ is any function and c is a constant, then

$$(c \cdot f(x))' = c \cdot f'(x).$$

Proof: We have

$$(c \cdot f(x))' = \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h}$$

$$= \lim_{h \to 0} c\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= c \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= c \cdot f'(x).$$

///

☆ Claim: If $f(x)$ and $g(x)$ are any functions then

$$(f(x) + g(x))' = f'(x) + g'(x).$$

Proof: Let $F(x) = f(x) + g(x)$. Then

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}$$

$$= \lim_{h \to 0} \frac{(f(x+h) - f(x)) + (g(x+h) - g(x))}{h}.$$

$$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right]$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x).$$ ///

Similarly we could show that

$$(f(x) - g(x))' = f'(x) - g'(x).$$

What other shortcuts do we know?

We previously computed

$$(x)' = 1$$
$$(x^2)' = 2x.$$

This is part of a pattern.

☆ The Power Rule: For any real number p we have

$$(x^p)' = p \cdot x^{p-1}.$$

Proof: We will only show this when $p = n$ is a positive integer. The general case is much harder.

$$(x^n)' = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

$$= \lim_{h \to 0} \frac{\cancel{x^n} + nx^{n-1}h + h^2(\text{something}) - \cancel{x^n}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(nx^{n-1} + h(\text{something}))}{\cancel{h}}$$

$$= n \cdot x^{n-1} + 0 \cdot (\text{something})$$

$$= n \cdot x^{n-1}.$$

At this point we can quickly differentiate any polynomial function:

Example: Let $f(x) = x^2 - 3x + 2$

$$f'(x) = (x^2 - 3x + 2)'$$

$$= (x^2)' - (3x)' + (2)'$$

$$= 2x - 3(x)' + 0$$

$$= 2x - 3.$$

And some simple radical functions:

Let $g(u) = \sqrt{u}\,(u+1)^2$

Compute $g'(u)$.

First we put it in recognizable form.

$$g(u) = \sqrt{u}\,(u+1)^2$$

$$= \sqrt{u}\,(u^2+2u+1)$$

$$= \sqrt{u}\cdot u^2 + 2\sqrt{u}\cdot u + \sqrt{u}.$$

$$= u^{1/2}\cdot u^2 + 2u^{1/2}\cdot u^1 + u^{1/2}$$

$$= u^{1/2+2} + 2u^{1/2+1} + u^{1/2}$$

$$= u^{5/2} + 2u^{3/2} + u^{1/2}.$$

Then $g'(u) = \frac{5}{2}u^{5/2-1} + 2\cdot\frac{3}{2}u^{3/2-1} + \frac{1}{2}u^{1/2-1}$

$$= \frac{5}{2}u^{3/2} + 3u^{1/2} + \frac{1}{2}u^{-1/2}. \quad ///$$

But some functions are still beyond our reach. For example,

$$f(x) = \sqrt{x^2+1}$$

We don't have a trick for this yet. Our only option right now is the Long way.

Practice: From Chapter 2.3 Exercises.

1. $f(x) = 2^{40}$.

5. $f(x) = x^3 - 4x + 6$

13. $A(s) = -12/s^5$

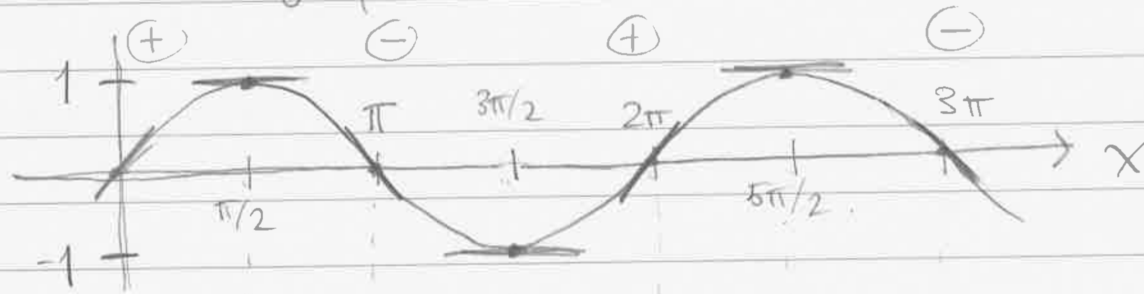
21. $v = t^2 - 1/\sqrt[4]{t^3}$

Before moving on to more powerful general-purpose tricks, let's think a bit about trigonometric functions.

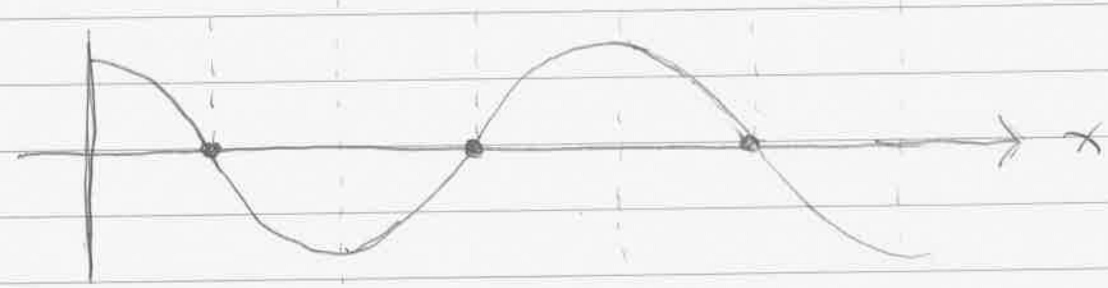
If $f(x) = \sin x$ then $f'(x) = ?$

Sketch the graph!

f



f'



Well, it kind of looks like

$$f'(x) = \cos(x).$$

Could that possibly be true?

Let's check:

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}.$$

The back of the book reminds us that

$$\sin(x+h) = \sin x \cos h - \cos x \sin h.$$

So,

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\left(\sin x \cos h + \cos x \sin h - \sin x\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\cos x \sin h + \sin x(\cos h - 1)\right]$$

$$= \lim_{h\to 0} \left[\cos x\left(\frac{\sin h}{h}\right) + \sin x\left(\frac{\cos h - 1}{h}\right)\right]$$

$$= \cos x \left(\lim_{h \to 0} \frac{\sin h}{h} \right) + \sin x \left(\lim_{h \to 0} \frac{\cos h - 1}{h} \right)$$

Luckily (!) we have seen one of these limits before. Recall that

$$\boxed{\lim_{h \to 0} \frac{\sin h}{h} = 1}$$

What about

$$\lim_{h \to 0} \frac{\cos h - 1}{h} \text{ ?}$$

Instead of a geometric argument we can just use an algebraic trick:

$$\lim_{h \to 0} \frac{\cos h - 1}{h} \left(\frac{\cos h + 1}{\cos h + 1} \right)$$

$$= \lim_{h \to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)}$$

$$= \lim_{h \to 0} \frac{-\sin^2 h}{h(\cos h + 1)}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} \cdot \frac{-\sin h}{\cos h + 1}.$$

$$= \lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \frac{-\sin h}{\cos h + 1}$$

$$= 1 \cdot \frac{0}{2}$$

$$= 0.$$

We conclude that $\boxed{\lim_{h \to 0} \frac{\cos h - 1}{h} = 0}$

and hence that

$$(\sin x)' = \cos x \left(\lim_{h \to 0} \frac{\sin h}{h} \right) + \sin x \left(\lim_{h \to 0} \frac{\cos h - 1}{h} \right)$$

$$= \cos x \cdot 1 + \sin x \cdot 0$$

$$= \cos x.$$

Sorry that was a bit long. This was the most complicated calculation we'll do in the class.

Using a similar argument we could show

$$(\cos x)' = -\sin x.$$

But we won't bother because we'll find a much better way to do it later.

Summary in Leibniz notation:

- $\frac{d}{dx} c = 0$
- $\frac{d}{dx} c \cdot f(x) = c \cdot \frac{d}{dx} f(x)$
- $\frac{d}{dx} (f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x).$
- $\frac{d}{dx} x^p = p \cdot x^{p-1}.$
- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$

///

That's a lot of good tricks, but the two most powerful tricks are still to come...

They are the PRODUCT and CHAIN rules. We'll discuss them tomorrow.

7/10/15

HW2 due NOW.
- look at solutions and discuss.

Quiz 2 on Monday.
- I'll post practice problems on the web, just like last time.

Yesterday we developed a bunch of tricks for computing derivatives. Unfortunately there are still plenty of functions we don't know how to differentiate, for example

$$f(x) = \sqrt{x^2+1}$$

$$g(u) = u \cdot \sin(u)$$

$$y = \tan(\theta)$$

Today we will develop two powerful tricks that will allow us to differentiate pretty much any function you can write down.

Then we'll practice using them.

Recall that for any functions $f(x)$ and $g(x)$ we can differentiate the sum and difference.

- $(f(x)+g(x))' = f'(x) + g'(x)$.
- $(f(x)-g(x))' = f'(x) - g'(x)$.

If you had to guess, how would you differentiate the product?

$$(f(x)\cdot g(x))' = ?$$

You would probably guess that

$$(f(x)\cdot g(x))' = f'(x)\cdot g'(x),$$

but this is WRONG!

For example, let $f(x) = x$ and $g(x) = x$. Then

$$f'(x)\cdot g'(x) = 1\cdot 1 = 1$$

but $(f(x)\cdot g(x))' = (x\cdot x)'$
$= (x^2)' = 2x$.

OK, so what is the correct rule?

☆ The Product Rule: Let $f(x)$ and $g(x)$ be any functions. Then we have

$$\boxed{(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)}$$

Let's test it on our example. If $f(x) = x$ and $g(x) = x$ then $f(x) \cdot g(x) = x^2$, and the product rule gives

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$= 1 \cdot x + x \cdot 1$$

$$= 2x. \quad \checkmark$$

So it works. But why does it work?

Please allow me to show you because it's a cute trick.

Proof: Consider any functions $f(x)$ and $g(x)$, and define $F(x) = f(x) \cdot g(x)$.

Then we have

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

Now what? We can add and subtract $f(x+h) \cdot g(x)$ from the numerator (that's the cute trick) to get

$$F'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \left[f(x+h) \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h} \right]$$

$$= \lim_{h \to 0} f(x+h) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + g(x) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

///

Now let's see what the product rule allows us to do.

Practice: Chapter 2.4 Exercises

1. Compute the derivative of $f(x) = (1+2x^2)(x-x^2)$ in two different ways.

3. $g(t) = t^3 \cdot \cos t$. Compute $g'(t)$.

1. First we'll do it without the product rule.

$$f(x) = (1+2x^2)(x-x^2)$$
$$= x - x^2 + 2x^3 - 2x^4$$

so
$$f'(x) = 1 - 2x + 2\cdot 3x^2 - 2\cdot 4x^3$$
$$= 1 - 2x + 6x^2 - 8x^3.$$

Now we'll use the product rule.

$$f'(x) = (1+2x^2)(x-x^2)' + (1+2x^2)'(x-x^2)$$
$$= (1+2x^2)(1-2x) + (4x)(x-x^2)$$
$$= 1 + \cancel{2x^2} - 2x - \cancel{4x^3} + \cancel{4x^2} - \cancel{4x^3}$$
$$= 1 - 2x + 6x^2 - 8x^3.$$

We got the same answer.

Often there are multiple ways to compute a derivative. You are free to use whichever method you want.

3. $g(t) = t^3 \cdot \cos t$.

Here the product rule is the only way to proceed. We have

$$g'(t) = t^3 \cdot (\cos t)' + (t^3)' \cdot \cos t$$

$$= t^3(-\sin t) + (3t^2) \cdot \cos t.$$

$$= -t^3 \cdot \sin t + 3t^2 \cdot \cos t.$$

OK, now what about the quotient of functions

$$\left(\frac{f(x)}{g(x)}\right)' = ?$$

Is there a "quotient rule"?

Yes.

☆ The Quotient Rule: Let $f(x)$ and $g(x)$ be any functions. Then we have

$$\boxed{\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g(x)^2}}$$

This one is a bit harder to remember but I'll show you later why you don't need to memorize it.

For now let's test it out.

Example: Compute $\frac{d}{dx}(\tan x)$.

Recall that we have

$$\tan x = \frac{\sin x}{\cos x}.$$

So we can use the quotient rule to get

$$\left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\,(\sin x)' - \sin x\,(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= 1/\cos^2 x.$$

In summary, we have

$$\frac{d}{dx}(\tan x) = 1/\cos^2 x.$$

At this point we can differentiate any trigonometric function.

Practice: Chapter 2.4 Exercises

13. $y = x^3/(1-x^2)$.

19. $y = x/(2-\tan x)$

13. Using the quotient rule gives

$$\frac{dy}{dx} = \frac{(1-x^2)\cdot(x^3)' - x^3\cdot(1-x^2)'}{(1-x^2)^2}$$

$$= \frac{(1-x^2)(3x^2) - x^3(-2x)}{(1-x^2)^2}$$

$$= \frac{3x^2 - 3x^4 + 2x^4}{(1-x^2)^2}$$

$$= (3x^2 - x^4) / (1-x^2)^2.$$

$$= x^2(3-x)/(1-x^2)^2.$$

19. Again the quotient rule.

$$\frac{dy}{dx} = \frac{(2-\tan x)(x)' - x(2-\tan x)'}{(2-\tan x)^2}$$

$$= \frac{2-\tan x - x(-1/\cos^2 x).}{(2-\tan x)^2}$$

Does it simplify? Probably not. //

Why did I say that you don't need to memorize the quotient rule?

Because it follows from the product rule and another rule that is easier to memorize.

Recall that apart from $+, -, \times,$ and $\div$, there is another way to combine two functions called composition.

Definition: Let $f(x)$ and $g(x)$ be any functions. Then we define a new function $(f \circ g)(x)$ by

$$(f \circ g)(x) = f(g(x)).$$

We call $f \circ g$ "f follows g" or "f composed with g". The basic idea is that we do g first, then do f.

Example: Let $f(x) = \sin(x)$ and $g(x) = x^2 + 1$. Then we have

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f(x^2+1) \\
&= \sin(x^2+1)
\end{aligned}
$$

and

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= f(x)^2 + 1 \\
&= \sin^2 x + 1 .
\end{aligned}
$$

[Remark : Note that in general

$$(f \circ g)(x) \neq (g \circ f)(x).$$]

The "chain rule" tells us how to differentiate the composition of two functions.

☆ The Chain Rule : Let $f(x)$ and $g(x)$ be any two functions. Then

$$
\boxed{
\begin{aligned}
(f \circ g)'(x) &= f'(g(x)) \cdot g'(x). \\
&= (f' \circ g)(x) \cdot g'(x)
\end{aligned}
}
$$

That looks a bit tricky, but it's much easier to remember in Leibniz notation.

Let y be a function of u (say $y = f(u)$) and let u be a function of x (say $u = g(x)$). Then y is also a function of x. Indeed, we have

$$y = f(u) = f(g(x)) = (f \circ g)(x).$$

We want to compute dy/dx, which is the same as $(f \circ g)'(x)$.

Note that $dy/du = f'(u) = f'(g(x))$ $= (f' \circ g)(x)$, and $du/dx = g'(x)$.

Then the chain rule

$$(f \circ g)'(x) = (f' \circ g)(x) \cdot g'(x)$$

gets translated into

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

And that just LOOKS TRUE!

☆ The Chain Rule in Leibniz Notation:

Let y be a function of u and let u be a function of x (so y is also a function of x). Then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

That's very easy to memorize but not very easy to prove, so we won't prove it.

Instead we'll learn how to use it.

Recall the function $f(x) = \sqrt{x^2+1}$.

We can express this as a composition of two functions. Let $g(x) = \sqrt{x} = x^{1/2}$ and let $h(x) = x^2+1$. Then

$$\begin{aligned} f(x) &= \sqrt{x^2+1} \\ &= g(x^2+1) \\ &= g(h(x)) = (g \circ h)(x). \end{aligned}$$

We compute $g'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

and $h'(x) = 2x$. Then the chain rule says

$$
\begin{aligned}
f'(x) &= (g \circ h)'(x) \\
&= (g' \circ h)(x) \cdot h'(x) \\
&= \frac{1}{2\sqrt{h(x)}} \cdot h'(x). \\
&= \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+1}}.
\end{aligned}
$$

But writing it that way makes it look hard. It's actually easier than that

$$f(x) = (x^2+1)^{1/2} = (\text{inside function})^{1/2}.$$

Then

$$
\begin{aligned}
f'(x) &= \frac{1}{2}(\text{inside function})^{-1/2} \cdot (\text{inside function})' \\
&= \frac{1}{2}(x^2+1)^{-1/2} \cdot (2x).
\end{aligned}
$$

///

Practice: Recall the example

$$f(x) = \sin(x) \quad \& \quad g(x) = x^2+1.$$

Compute $(f \circ g)'(x)$ and $(g \circ f)'(x)$.

- $(f \circ g)(x) = \sin(x^2+1)$.

We can just think of this as

$$\sin(\text{something}).$$

Then $(f \circ g)'(x)$ is

$$(\sin(\text{something}))' = \cos(\text{something}) \cdot (\text{something})'$$
$$= \cos(x^2+1)(2x)$$

- $(g \circ f)(x) = (\sin x)^2 + 1$
$$= (\text{something})^2 + 1.$$

The derivative is

$$((\text{something})^2+1)' = (2(\text{something})) \cdot (\text{something})'$$
$$= (2\sin x) \cdot (\cos x).$$

For Quiz 2 :

- Look at HW 2 solutions.

- Product Rule <u>is</u> on the quiz.

- Quotient & Chain rules are <u>not</u> on the quiz. (We need more time to absorb them.)

- <u>I</u>'ll post practice problems on the web (from 2nd edition of Stewart).