6/29/15

Welcome to Calculus I
MTH 161 - QR
MTWRF 10:05 - 12:05 Memorial 312

Textbook: Essential Calculus by
James Stewart, 2nd edition.

We will cover all of Chapters 1-4
and some of Chapters 5-7.

For additional reading I recommend

"The Hitchhiker's Guide to Calculus"
by Michael Spivak.

Course Details:

I am Drew Armstrong. All course materials will be posted on my webpage:

www.math.miami.edu/narmstrong/161sum15

Currently I have no office, so we'll have to get creative about office hours.
(stay tuned)

I can be reached via email:

armstrong@math.miami.edu.

Your grade in this course will be based on 3 components:

1/3 Homework
1/3 In-Class Quizzes
1/3 Final Exam.

I will assign homework weekly, to be turned in at the beginning of each Friday's class.

There will be a Quiz at the beginning of class each Monday.

I will lecture for part of each class and the rest will be group work and practicing example problems together. The balance of class time is roughly

3/4 Lecture, 1/4 Recitation.

Let's Begin.

Stewart's text contains enough material for 3 semesters:

- Calculus I
- Calculus II
- Calculus III / Multivariable or Vector Calculus.

So Calculus is a big subject. But what is it all about?

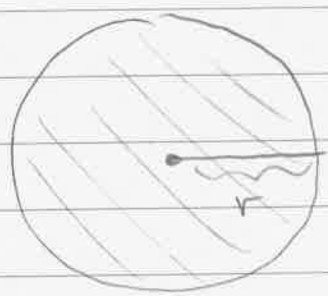
"Calculus" literally means a small pebble used for calculating on an abacus.

So Calculus has something to do with calculations. Actually, it is a collection of techniques invented in the 1660-70s that turned whole families of previously difficult problems into routine calculations. This paved the way for huge advances in theoretical and applied science.

To illustrate the kinds of difficult problems that Calculus allows us to solve, today we'll go back to the beginning.

Historical Introduction:

One could say that the oldest theorem of Calculus is the formula for the area of a circle.



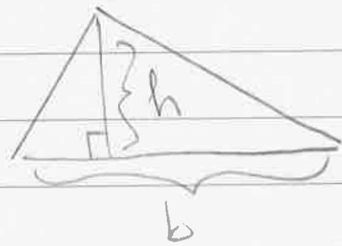
The area is

$$\pi \cdot r^2.$$

radius r.

We all know this formula, but <u>why</u> is it true? The formula was explained by Archimedes of Syracuse (c. 287–212 BC). We will follow his method.

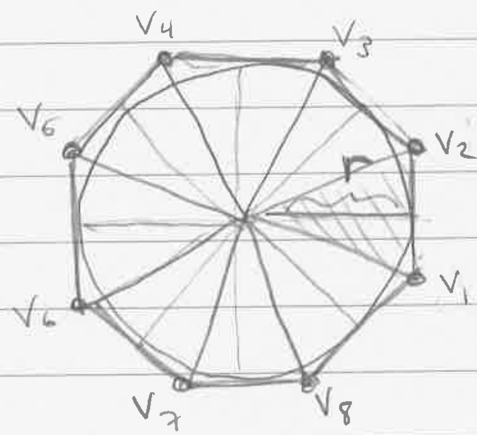
Recall that the area of a triangle is
$\frac{1}{2} \cdot \text{base} \cdot \text{height}$:



$$\text{area} = \frac{1}{2} h b .$$

[Exercise: Why is <u>this</u> formula true?]

Archimedes' idea is to enclose (inscribe) the circle in a regular polygon P.



Here P is an octogon. The area of the shaded triangle is

$$\frac{1}{2} \cdot r \cdot \text{length of } V_1 V_2$$

The next triangle has area

$$\frac{1}{2} r \cdot \text{length of } V_2V_3 .$$

Adding all eight triangles gives

(*) $$\text{area of } P = \frac{1}{2} \cdot r \cdot \text{perimeter of } P .$$

But the same result will hold for any regular polygon. As we choose polygons with a greater number of sides. The area of P approaches the area of the circle, and the perimeter of P approaches the circumference of the circle, i.e., $2\pi r$.

Thus the equation (*) approaches

$$\begin{aligned} \text{area of circle} &= \frac{1}{\cancel{2}} \cdot r \cdot \cancel{2}\pi r \\ &= \pi \cdot r \cdot r \\ &= \pi r^2 \end{aligned}$$

///

The strange part of this argument is that it involves the idea of infinity.

Let P_n be the regular polygon with n sides circumscribing a circle C of radius r. Then we have

$$(*) \quad \text{area of } P_n = \frac{1}{2} \cdot r \cdot \text{perimeter of } P_n .$$

and the perimeter of P_n approaches $2\pi r$ as n approaches infinity (∞).

In modern language we would write

$$\lim_{n \to \infty} (\text{perimeter of } P_n) = 2\pi r .$$

The symbol "$\lim\limits_{n\to\infty}$" means "the limit as n approaches ∞".

The rest of Archimedes' proof could be written using this notation as follows:

$$\text{area of } C = \lim_{n\to\infty} \left(\text{area of } P_n \right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{2} \cdot r \cdot \text{perimeter of } P_n \right) \qquad \text{from } (*)$$

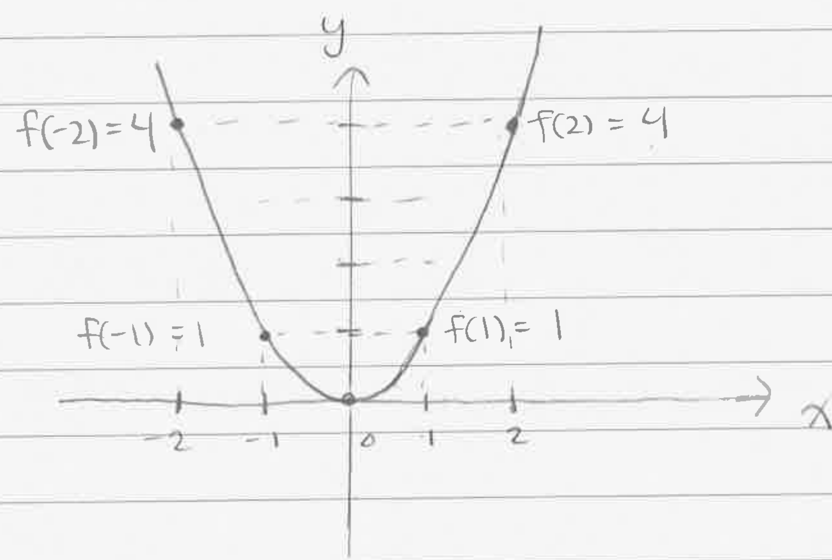
$$= \frac{1}{2} \cdot r \cdot \lim_{n\to\infty} \left(\text{perimeter of } P_n \right)$$

$$= \frac{1}{2} \cdot r \cdot 2\pi r = \pi r^2 .$$

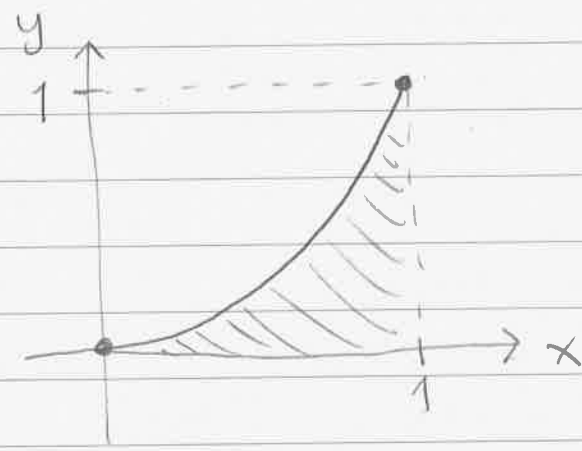
///

OK, fine. Now let's try to compute the area of a harder shape:

Recall that the graph of the function $f(x) = x^2$ is called a parabola



Now consider the region between the parabola and the x-axis from $x=0$ to $x=1$:



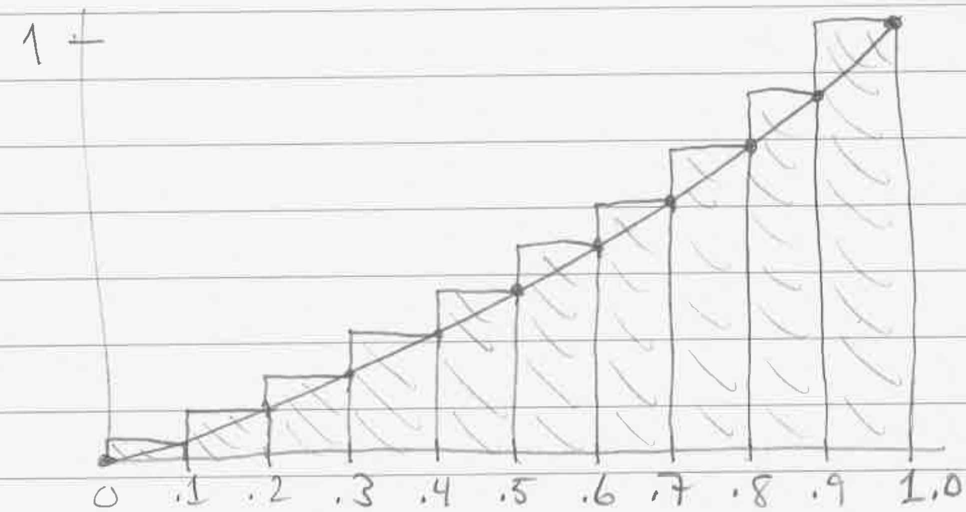
What is the area of this region?

Later, the techniques of Calculus will allow us to write down the answer almost immediately.

But Archimedes knew the answer to this problem, so it must be possible to do it without Calculus.

Let's try. We'll use the idea of approximating our region with an easier region whose area we know.

For example, let's approximate our region with 10 rectangles



The width of each rectangle is 0.1 and the heights of the rectangles are $(.1)^2, (.2)^2, \ldots, (.9)^2, (1.0)^2$.

Thus the total area is

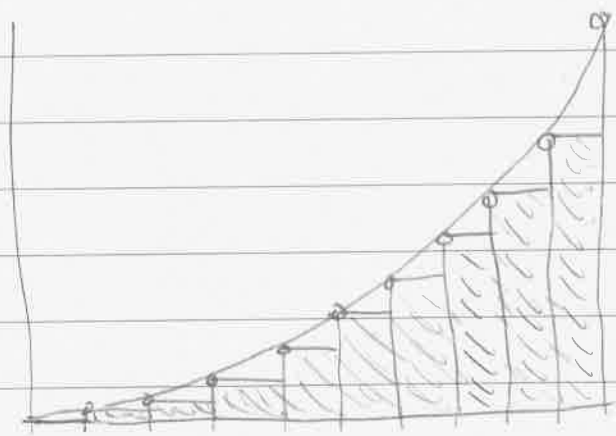
$$(.1)(.1)^2 + (.1)(.2)^2 + \cdots + (.1)(.9)^2 + (.1)(1.0)^2$$

$$= (.1)\left[(.1)^2 + (.2)^2 + \cdots + (.9)^2 + (1.0)^2\right]$$

$$= 0.385.$$

That's an over-estimate because parts of the rectangles extend above the parabola.

We can get an under-estimate by using the left endpoints to compute the height of each rectangle.



The under-estimate is

$$(.1)(0)^2 + (.1)(.1)^2 + \cdots + (.1)(.8)^2 + (.1)(.9)^2$$

$$= (.1)\left[0^2 + (.1)^2 + (.2)^2 + \cdots + (.8)^2 + (.9)^2\right]$$

$$= 0.285.$$

So we know that the area of our region is between 0.285 and 0.385.

Maybe the average is a good estimate.

$$\frac{0.285 + 0.385}{2} = 0.335$$

But we want the <u>exact</u> answer.
With enough calculations, you could convince yourself that the area is probably $0.3333\cdots = 1/3$.

We'll see tomorrow <u>why</u> this is true.

6/30/15

Calculus I
MTH 161 - QR

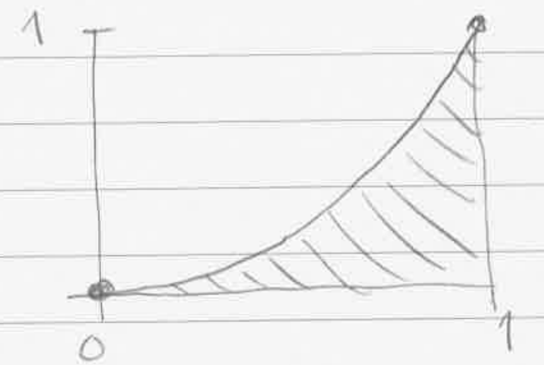
Office Hours still TBA. (Since I don't have an office this is a bit tricky).

Before we dive into a systematic development (Stewart, Chap. 1-4), I am giving a historical introduction to the main ideas of Calculus.

For this reason HW1 (due at the beginning of class Monday) is a bit strange. In particular, it doesn't use any problems from the textbook.

So, where were we? We were trying to compute the area of a tricky shape.

Consider the region between the graph of $f(x) = x^2$ and the x-axis, from $x = 0$ to $x = 1$.



By approximating this region with 10 rectangles, we decided that the area is slightly less than 0.335.

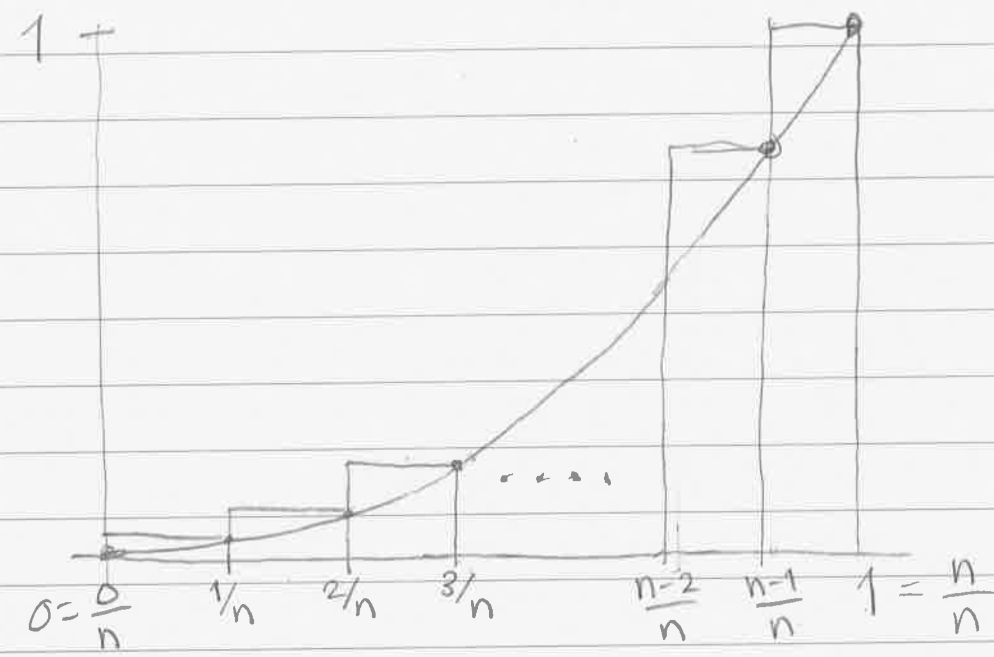
To get a more accurate estimate we can use more rectangles. The more rectangles, the better the estimate. To get the exact answer we will let the number of rectangles approach ∞.

What?! How?

To do this we need to be a bit abstract. We will divide the interval from $x = 0$ to $x = 1$ into n equal intervals of length $1/n$.

↓

It's a bit hard to draw, but here's the idea:



The height of the i^{th} triangle is $\left(\frac{i}{n}\right)^2$, so the total area of the rectangles is

$$\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2 + \left(\frac{n}{n}\right)^2\right]$$

$$= \frac{1}{n}\left[\frac{1^2}{n} + \frac{2^2}{n^2} + \frac{3^2}{n^2} + \cdots + \frac{(n-1)^2}{n^2} + \frac{n^2}{n^2}\right]$$

$$= \frac{1}{n} \cdot \frac{1}{n^2} \left[1^2 + 2^2 + 3^2 + \cdots + n^2 \right]$$

$$= \frac{1}{n^3} \left[1^2 + 2^2 + 3^2 + \cdots + n^2 \right].$$

This is the area of the n rectangles. The exact area our region should be the limit of this expression as n approaches ∞.

$$\text{area} = \lim_{n \to \infty} \frac{1}{n^3} \left[1^2 + 2^2 + \cdots + n^2 \right]$$

Ok, fine. But how do we compute this limit? We will need some algebraic tricks. In this case we can use a very tricky formula:

$$(*) \quad 1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$$

$$= \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n$$

[Don't worry right now about why (*) is true. We'll just use it.]

Thus,

$$\text{area} = \lim_{n\to\infty} \frac{1}{n^3}\left[1^2+2^2+\cdots+n^2\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left[\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right]$$

NOW, can we compute the limit?

Yes! Because we know that

$$\frac{1}{2n} \to 0 \quad \text{and} \quad \frac{1}{6n^2} \to 0$$

as $n \to \infty$.

Hence,

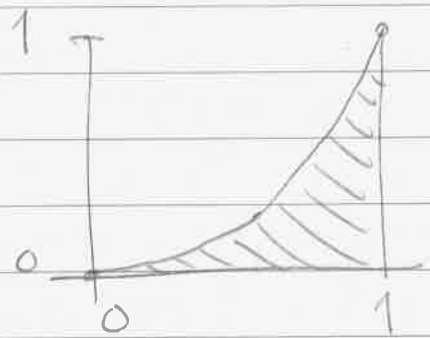
$$\text{area} = \lim_{n\to\infty} \left[\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{3} + \underbrace{\cancel{\lim_{n\to\infty}\frac{1}{2n}}}_{0} + \underbrace{\cancel{\lim_{n\to\infty}\frac{1}{6n^2}}}_{0}$$

$$= \lim_{n \to \infty} \frac{1}{3} = \frac{1}{3}$$

because $1/3$ is constant with respect to n.

Finally, we have the exact answer.



area $= 1/3$.

[Remark: This exact calculation can be found on pg 201 of Stewart and pg 92 of Spivak.]

Wow, that was pretty tricky.
Why did I show it to you now?

Two reasons:

Reason 1 : Because this kind of problem was the historical origin of Calculus.

Reason 2 : Because we will learn a much more efficient method later. You will appreciate the efficient method more if you first see how hard it used to be.

While we're at it, here is some modern notation. We use the expression

$$\int_0^1 x^2\,dx$$

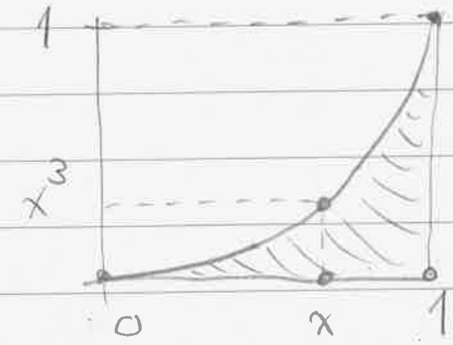
to denote the <u>area</u> under the graph of x^2 from $x=0$ to $x=1$. The symbol $\int$ is a stretched letter "S". It was introduced in the 1670s by Gottfried Wilhelm Leibniz and he called the expression

$$\int_0^1 x^2\,dx \quad \text{on "integral".}$$

The main goal of Calculus is to compute integrals of arbitrary functions.

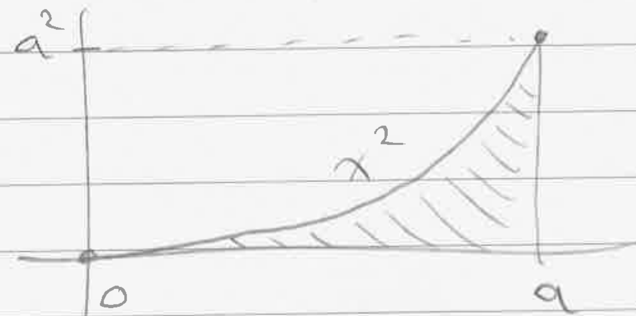
On HW1 you will use a similar method to show that

$$\int_0^1 x^3\, dx = \frac{1}{4}$$



[Don't worry. I'll supply the necessary algebraic tricks.]

To finish the story let me note that one could more generally compute the area from $x=0$ to $x=a$ for any a.



The same method we used will tell you that the area is

$$\int_0^a x^2\,dx = \frac{a^3}{3}$$

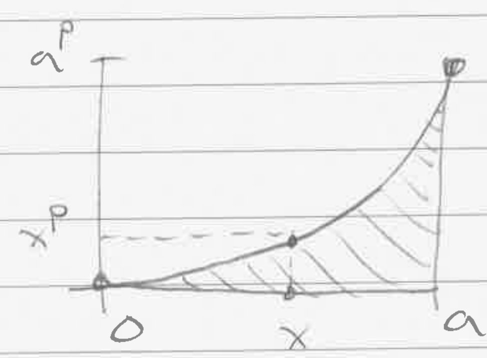
and the same method you will use on HW1 can be used to show that

$$\int_0^a x^3\,dx = \frac{a^4}{4}.$$

Can you see a pattern here?

In 1636, Pierre de Fermat showed that for all powers $p \neq -1$ we have

$$\int_0^a x^p\,dx = \frac{a^{p+1}}{p+1}$$



This was a major result because it led Newton & Leibniz in the 1660s and 1670s to discover the Fundamental Theorem of Calculus.

Anyway, more about that later. Before discussing the F.T.C. we must go back to the beginning and develop things more systematically.

End of introduction.
Tomorrow we'll discuss Chapter 1

7/1/15

Now that we have the historical introduction under our belts, let's begin the systematic development of Calculus.

My goal is to cover <u>all</u> of the ideas in Stewart Chap 1–4 and <u>some</u> of the ideas in Chap. 5–7.

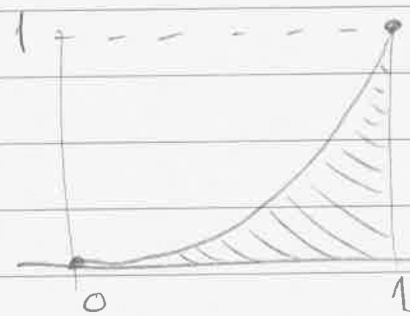
Chapter 1: Functions & Limits.

The most basic concept in Calculus is the idea of a <u>Limit</u>.

We have already encountered limits when we computed the areas of circles and parabolic regions.

Recall the final step from our calculation of the area under a parabola.

$$\int_0^1 x^2\,dx =$$



the shaded area.

After some work we found that

$$\int_0^1 x^2\,dx = \lim_{n\to\infty}\left[\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{3} + \lim_{n\to\infty}\frac{1}{2n} + \lim_{n\to\infty}\frac{1}{6n^2}$$

$$= \frac{1}{3} + 0 + 0 = \frac{1}{3}.$$

This kind of calculation will become a routine for us.

If $f(n)$ is any function of an integer ("whole number") variable n then we can consider the limit of $f(n)$ as n approaches ∞.

We write

$$\boxed{\lim_{n\to\infty} f(n) = L}$$

to mean that "there exists a number L such that we can make $f(n)$ arbitrarily close to L by by taking n to be arbitrarily large."

For example, you know that

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

because we can make $1/n$ "as close to 0 as we want" by making n "big enough".

Example: Compute

$$\lim_{n \to \infty} \frac{n}{n+1}$$

Solution: Let's do a trick. We can divide the numerator and denominator both by n without changing the value of the function.

$$\lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{n/n}{(n+1)/n}$$

$$= \lim_{n \to \infty} \frac{1}{1+\frac{1}{n}}.$$

Then we are allowed to bring the limits inside:

$$\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}}$$

$$= \frac{1}{1+0} = 1.$$

Exercise: Try this one

$$\lim_{n\to\infty} \frac{5n^2+3}{2n^2+n} = ?$$

[Hint: Divide the numerator and denominator by something.]

WARNING: Limits might not exist.

Example: $\lim_{n\to\infty} n = ?$

Does not exist! But we might use the convenient notation

$$\text{“}\lim_{n\to\infty} n = \infty\text{”}$$

as long as we agree not to take it too literally. [∞ is not a number. If you treat it like one you will make lots of mistakes.]

Example: $\displaystyle\lim_{n\to\infty} (-1)^n = ?$

Does not exist! Let $f(n) = (-1)^n$ and make a table of values

n	1	2	3	4	5	6	...
$f(n)$	-1	1	-1	1	-1	1	...

Note that $f(n)$ is not <u>approaching</u> any number; it's just bouncing back and forth. We don't even have a good shorthand notation for this.

Example: On HW1 you will consider

$$\lim_{n\to\infty} n\cdot\sin\left(\frac{\pi}{n}\right)$$

If you try to bring the limit inside, you get

$$\lim_{n \to \infty} n \cdot \sin\left(\frac{\pi}{n}\right).$$

$$= \lim_{n \to \infty} n \cdot \lim_{n \to \infty} \sin\left(\frac{\pi}{n}\right).$$

$$\stackrel{?}{=} \infty \cdot 0 \quad ?$$

This is not helpful. We are not allowed to treat ∞ as a number.

We need a better trick to compute this limit. Stay tuned. ///

For the record let me list some algebraic properties of limits.

Let $f(n)$ and $g(n)$ be functions and suppose that the limits

$$\lim_{n \to \infty} f(n) \quad \text{and} \quad \lim_{n \to \infty} g(n)$$

<u>exist</u> (and are <u>finite</u>).

Then we have

- $\lim_{n \to \infty} (f(n) + g(n)) = \lim_{n \to \infty} f(n) + \lim_{n \to \infty} g(n)$

- $\lim_{n \to \infty} (f(n) - g(n)) = \lim_{n \to \infty} f(n) - \lim_{n \to \infty} g(n)$

- $\lim_{n \to \infty} (c \cdot f(n)) = c \cdot \lim_{n \to \infty} f(n)$

  for any constant c.

- $\lim_{n \to \infty} (f(n) \cdot g(n)) = \lim_{n \to \infty} f(n) \cdot \lim_{n \to \infty} g(n)$.

- $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} f(n) \Big/ \lim_{n \to \infty} g(n)$

  [as long as $\lim_{n \to \infty} g(n) \neq 0$].

- $\lim_{n \to \infty} c = c$ for constant c.

With these rules we can compute quite a few limits.

However, the most interesting/difficult limits have the form

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty.$$

These are called "indeterminate forms". If we see one of these then we know there is more work to be done, and some kind of trick is probably needed.

Recitation: Evaluate the following limits if they exist.

1. $\lim\limits_{n \to \infty} (-1)^n \frac{1}{n}$

4. $\lim\limits_{n \to \infty} \sin(2\pi n)$

2. $\lim\limits_{n \to \infty} \frac{n^3}{n^2+1}$

5. $\lim\limits_{n \to \infty} \cos(\pi n)$

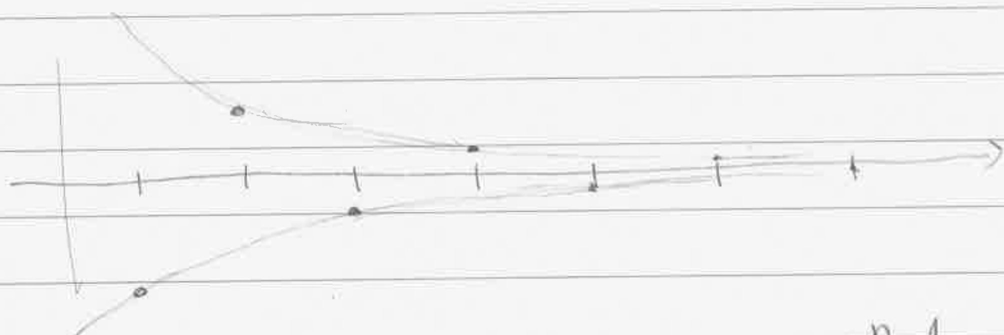
3. $\lim\limits_{n \to \infty} \frac{3^{n+2}}{5^n}$

6. $\lim\limits_{n \to \infty} \frac{n^2}{\sqrt{n^4+1}}$

1. Consider a table of values

n	1	2	3	4	5	6
$(-1)^n \frac{1}{n}$	-1	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{5}$	$\frac{1}{6}$

These values are bouncing back and forth, but ultimately they are approaching zero.



So we conclude that $\lim\limits_{n \to \infty} (-1)^n \frac{1}{n} = 0$.

2. Multiply top and bottom by $1/n^2$

$$\lim_{n \to \infty} \frac{n^3}{(n^2+1)} \cdot \frac{1/n^2}{1/n^2} = \lim_{n \to \infty} \frac{n}{1 + \frac{1}{n^2}}.$$

Numerator $n \longrightarrow \infty$

Denominator $1 + \frac{1}{n^2} \longrightarrow 1$

So $n / \cdot (1 + \frac{1}{n^2})$ is going to ∞.

We say $\lim_{n \to \infty} \frac{n^3}{n^2+1}$ does not exist.

3. $\lim_{n \to \infty} \frac{3^{n+2}}{5^n}$.

Simplify $\frac{3^{n+2}}{5^n} = \frac{3^n \cdot 3^2}{5^n} = 9 \cdot \frac{3^n}{5^n}$

$$= 9 \cdot \left(\frac{3}{5}\right)^n = 9 \cdot (0.6)^n$$

Note that $\lim_{n \to \infty} (0.6)^n \to 0$.

[Useful Fact : If $-1 < a < 1$ then

$$\lim_{n \to \infty} a^n = 0.$$]

Hence

$$\lim_{n \to \infty} \frac{3^{n+2}}{5^n} = \lim_{n \to \infty} 9 \cdot (0.6)^n$$

$$= 9 \cdot \lim_{n \to \infty} (0.6)^n$$

$$= 9 \cdot 0 = 0.$$

4. $\lim_{n \to \infty} \sin(2\pi n)$

This is a trick question because $\sin(2\pi n) = 0$ for any whole number n. Hence our limit is

$$\lim_{n \to \infty} \sin(2\pi n) = \lim_{n \to \infty} 0 = 0.$$

5. $\lim_{n \to \infty} \cos(\pi n)$

Table of values

n	1	2	3	4	5	6	...
$\cos(\pi n)$	-1	1	-1	1	-1	1	...

Note that $\cos(\pi n) = (-1)^n$.
The limit does not exist.

6. $\lim_{n \to \infty} \dfrac{n^2}{\sqrt{n^4+1}}$

Multiply top and bottom by $1/n^2$ to get

$$\lim_{n\to\infty} \frac{n^2}{\sqrt{n^4+1}} \cdot \frac{1/n^2}{1/n^2} = \lim_{n\to\infty} \frac{1}{\frac{1}{n^2}\sqrt{n^4+1}}.$$

Rewrite $1/n^2$ as $\sqrt{1/n^4}$ to get

$$\lim_{n\to\infty} \frac{1}{\sqrt{\frac{1}{n^4}}\sqrt{n^4+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{1}{n^4}(n^4+1)}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^4}}} = \frac{1}{\sqrt{1+0}} = 1.$$

[Remark: In the last step we did something like

$$\lim_{n\to\infty} \sqrt{f(n)} = \sqrt{\lim_{n\to\infty} f(n)}.$$

Is this allowed? Yes. The reason is because the square root is a "continuous function". We will see later that we are allowed to bring a limit inside of a continuous function.]

7/2/15

We are discussing Chapter 1, "Functions & Limits". (Although last time I snuck in some ideas from Chapter 8.1 on "Sequences").

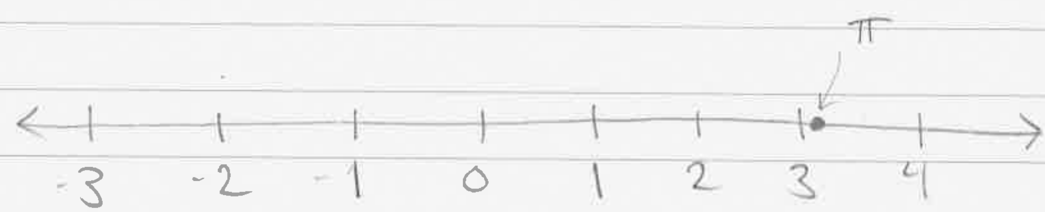
Last time we discussed the notation

$$\lim_{n \to \infty} f(n) = L$$

when $f(n)$ is a function of an <u>integer</u> (i.e. whole number) variable n.

Today we are interested in functions $f(x)$ of a <u>real</u> variable x.

Remark: A real number is any point on the number line



Real numbers have decimal expansions that may be finite, infinite repeating, or infinite nonrepeating.

For example,

$$1/8 = 0.125$$

$$1/3 = 0.3333\cdots$$

$$\pi = 3.14159265\cdots$$

are all real numbers.]

Functions can be described by formulas, for example,

$$f(x) = x^2$$

$$g(t) = e^{t^2+5}$$

$$h(s) = \sin\left(\frac{7}{\sqrt{\log(s)+2}}\right).$$

But they don't have to be. For example,

$f(t)$ = the distance of my car from $0°$ Latitude & $0°$ Longitude at time t, measured in meters, where $t = 0$ at 12:00 am on Jan 1, 2015.

This is certainly a function and it certainly doesn't have a nice algebraic formula.

A function $f(x)$ of a real variable x can be visualized by drawing its graph.

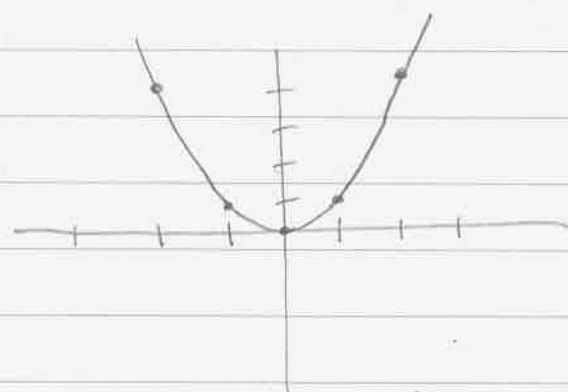
This is the collection of points

$$(x, f(x))$$

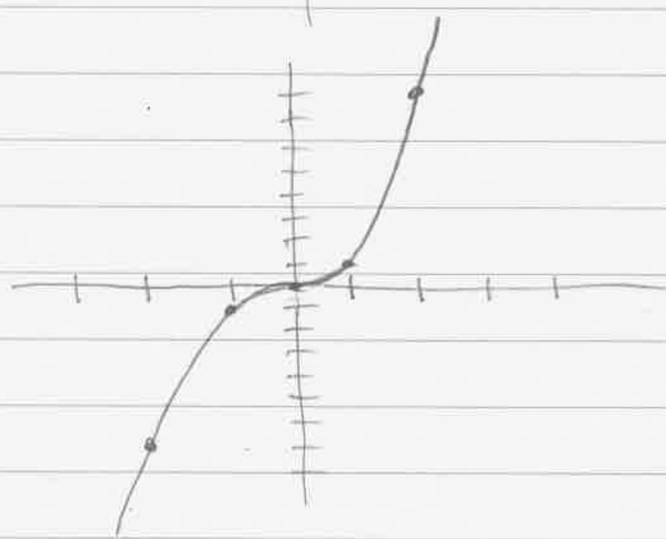
in the (x,y)-plane.

Examples:

The graph of $f(x) = x^2$



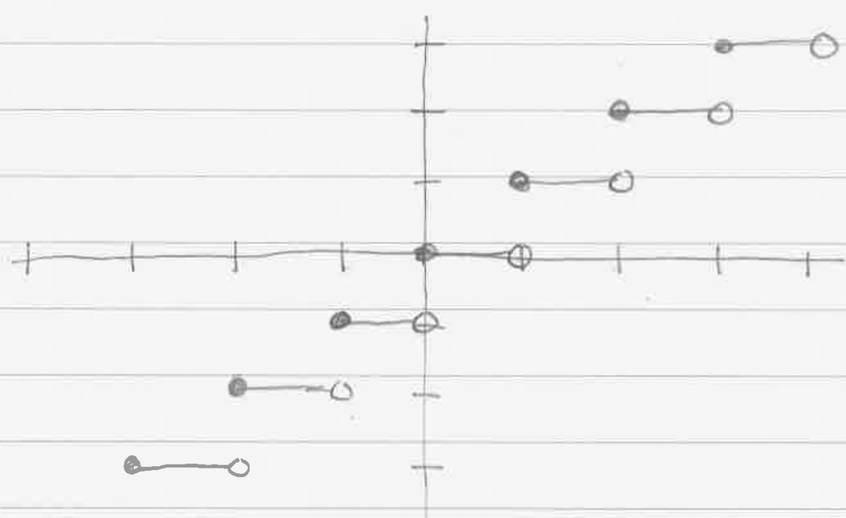
The graph of $f(x) = x^3$.



Here's a stranger kind of function. Given a real number x we define

$$\lfloor x \rfloor = \text{the greatest integer less than or equal to } x$$

The function $f(x) = \lfloor x \rfloor$ is called the floor function. Here is its graph:



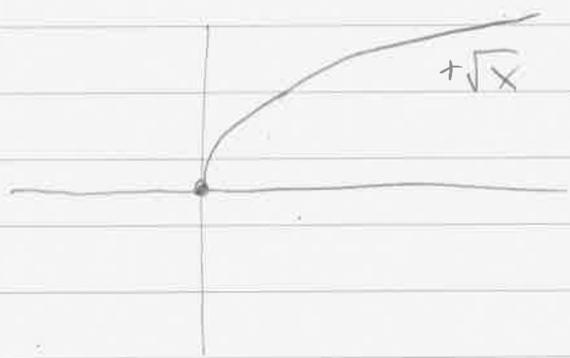
The closed and open dots are used to indicate that $f(n) = \lfloor n \rfloor = n$ when n is an integer.

Exercise: Draw the graph of the function $f(x) = \sqrt{x}$.

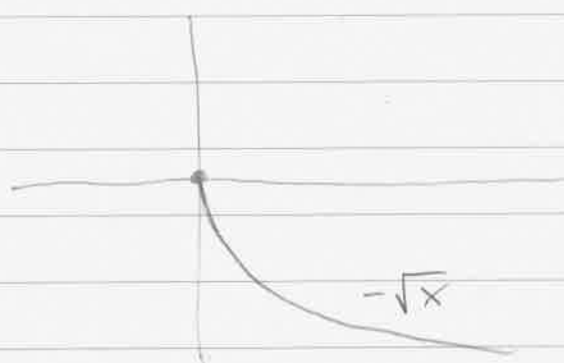
This was a trick question. We have to be careful because there is no such thing as <u>the</u> square root of x.

If $x > 0$ then it has two distinct square roots. Call them $+\sqrt{x}$ and $-\sqrt{x}$.

Then $g(x) = +\sqrt{x}$ is a function



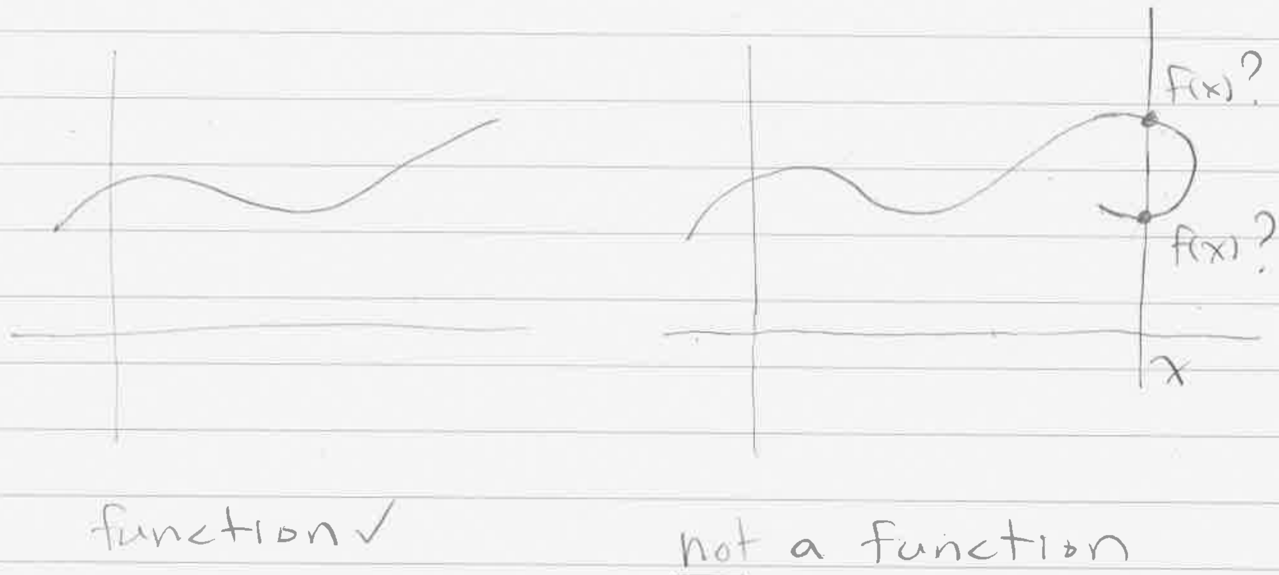
and $h(x) = -\sqrt{x}$ is a function



But $f(x) = \sqrt{x}$ is not really a function unless we indicate which square root we mean.

The Vertical Line Test:

Consider a curve in the (x,y)-plane. If some vertical line meets the curve in more than one point then this curve is <u>not</u> the graph of a function $f(x)$

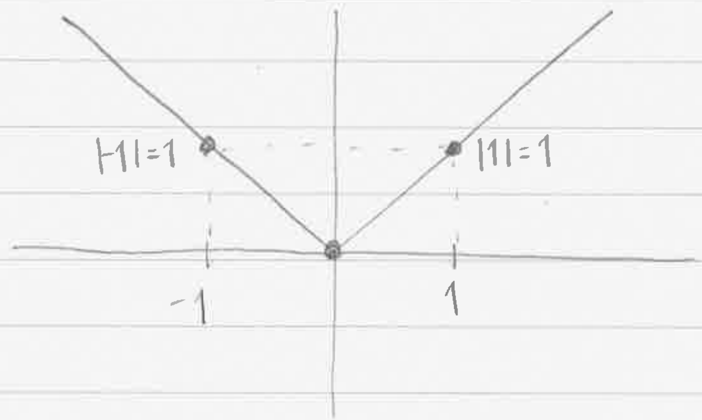


We can also define a function by different rules in different regions. This is called a piecewise defined function.

For example, consider the absolute value function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

The graph looks like this.



With functions of a real variable x we can still consider the limit as $x \to \infty$.

But now we can also consider the limit of $f(x)$ as $x \to a$ for some finite number a.

The notation

$$\boxed{\lim_{x \to a} f(x) = L}$$

means that "there exists a number L such that we can make $f(x)$ as close as we want to L by taking x sufficiently close to a."

Example:

$$\lim_{x \to 2} x^2 = 2^2 = 4$$

That was an easy one. We just plugged in $x = 2$.

It's not always so easy.

Example: Compute

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

Now we can't just plug in $x = 1$ because this gives us the "indeterminate form" $0/0$. What can we do?

— Try values of x near 1. Let $f(x) = \frac{x^2-1}{x-1}$.

$f(1.1) = 2.1$ $\quad$ $f(1.01) = 2.01$

$f(0.9) = 1.9$ $\quad$ $f(0.99) = 1.99$

Any guesses? It looks like $f(x) \to 2$ as $x \to 1$.

— Algebraic manipulation?

Recall the difference of squares formula:

$$a^2 - b^2 = (a-b)(a+b)$$

In particular, we have

$$x^2 - 1 = (x-1)(x+1)$$

Hence we have

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$$

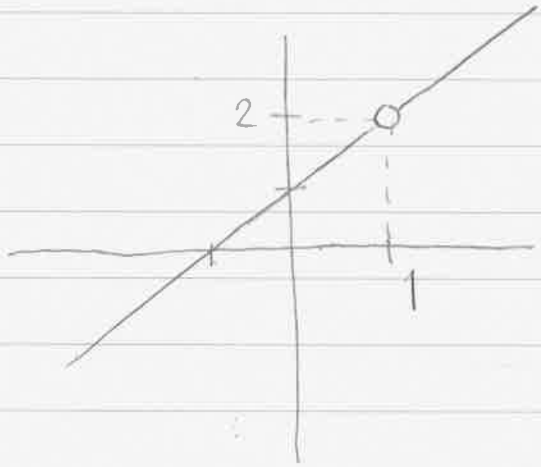
$$= \lim_{x \to 1} x+1 = 2.$$

Problem: Define $f(x) = \frac{x^2-1}{x-1}$, $g(x) = x+1$.

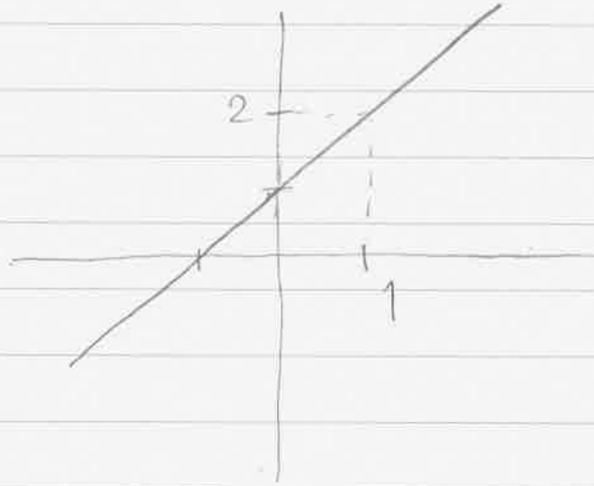
Are these the same function?

NOT QUITE. If $x \neq 1$ then we have $f(x) = g(x)$, but note that $g(1) = 2$ and $f(1)$ is <u>not</u> <u>defined</u>.

It's a subtle but important difference.



graph of f



graph of g.

We use the open dot to indicate that $f(x)$ is not defined when $x = 1$.

For the record, the table of algebraic properties of limits from last class is still true for limits of the form

$$\lim_{x \to a} f(x).$$

[See pg. 35 of Stewart.]

Recitation:

1. Does $\lim_{x \to 2} \lfloor x \rfloor$ exist?

2. $\lim_{x \to 5} \frac{x^2 - 6x + 5}{x - 5}$

3. $\lim_{h \to 0} \frac{+\sqrt{9+h} - 3}{h}$

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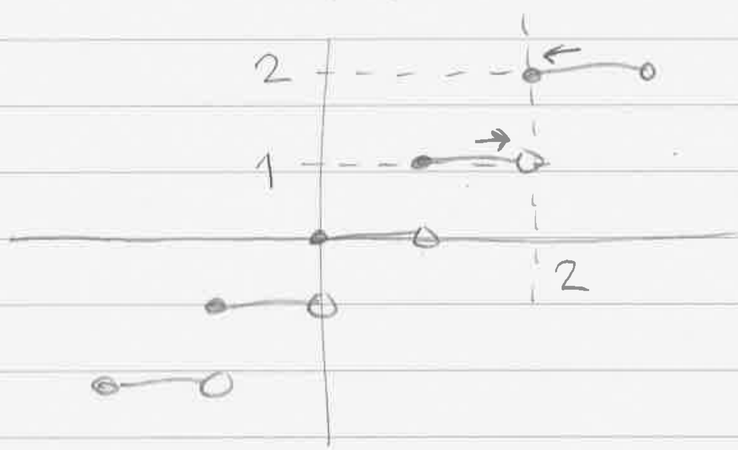
1. Recall the graph of $f(x) = \lfloor x \rfloor$



As x approaches 2 from the right, $f(x)$ approaches (in fact equals) 2.
As x approaches 2 from the left, $f(x)$ approaches (in fact equals) 1.
[The actual value of $f(2)$ is not relevant to the limit.].

Since f(x) approaches two different values as $x \to 2$ we say the limit

$$\lim_{x \to 2} \lfloor x \rfloor \text{ does not exist.}$$

We could be more specific by saying

$$\underbrace{\lim_{x \to 2^-}}_{\substack{\text{approaching} \\ \text{2 from the left}}} \lfloor x \rfloor = 1 \quad \& \quad \underbrace{\lim_{x \to 2^+}}_{\substack{\text{approaching} \\ \text{2 from the right}}} \lfloor x \rfloor = 2$$

[General Rule: We say that the limit

$$\lim_{x \to a} f(x)$$

exists if the two one-sided limits

$$\lim_{x \to a^-} f(x) \quad \& \quad \lim_{x \to a^+} f(x)$$

both exist and are equal to each-other.]

2. This limit has the indeterminate form 0/0, which gives us NO INFORMATION. So we need a trick.

Factor the numerator

$$x^2 - 6x + 5 = (x-1)(x-5)$$

to get

$$\lim_{x \to 5} \frac{x^2-6x+5}{x-5} = \lim_{x \to 5} \frac{(x-1)\cancel{(x-5)}}{\cancel{(x-5)}}$$

$$= \lim_{x \to 5} (x-1)$$

$$= 4.$$

3. This limit has indeterminate form 0/0 which gives NO INFORMATION. So we need a trick.

This time the trick is a bit trickier.

We can "rationalize the numerator" by multiplying top and bottom by the "conjugate" expression

$$\sqrt{9+h} + 3$$

Then we have

$$\lim_{h\to 0} \frac{\sqrt{9+h} - 3}{h} = \lim_{h\to 0} \frac{\left(\sqrt{9+h} - 3\right)}{h} \cdot \frac{\left(\sqrt{9+h} + 3\right)}{\left(\sqrt{9+h} + 3\right)}.$$

$$= \lim_{h\to 0} \frac{(\cancel{9}+h) - \cancel{9}}{h\left(\sqrt{9+h} + 3\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{9+h} + 3\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{9+h} + 3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$$